

Review exercises 2 Differential calculus, integral calculus

Problems

R2.1 Decide whether the statements below are true or false:

- a) "The derivative (derived function) of a function is a function."
- b) "The derivative (rate of change) of a function at a particular position is a number."
- c) "The function f has a local maximum at $x = x_1$ if $f'(x_1) = 0$ and $f''(x_1) > 0$."
- d) "If $f''(x_2) = 0$ and $f'''(x_2) < 0$, then the function f has a point of inflection at $x = x_2$."
- e) "If $g' = f$, then g is an antiderivative of f ."
- f) "f with $f(x) = 2x + 20$ is an antiderivative of g with $g(x) = x^2$."
- g) "f with $f(x) = 3x$ has infinitely many antiderivatives."
- h) "The indefinite integral of a function is a set of functions."

R2.2 Determine the value $f(x_0)$, the first derivative $f'(x_0)$, and the second derivative $f''(x_0)$ of the function f at the position x_0 :

- a) $f(x) = 4x^2(x^2 - 1)$ $x_0 = -1$
- b) $f(x) = (-3x^2 + 2x - 1) \cdot e^x$ $x_0 = -2$
- c) $f(x) = (x^2 + 2) \cdot e^{-3x}$ $x_0 = -\frac{1}{3}$

R2.3 For the given cost function $C(x)$ and revenue function $R(x)$ determine ...

- i) ... the marginal cost function $C'(x)$.
- ii) ... the marginal revenue function $R'(x)$.
- iii) ... the marginal profit function $P'(x)$.
- a) $C(x) = (40x + 200)$ CHF $R(x) = 60x$ CHF
- b) $C(x) = (5x^2 + 20x + 100)$ CHF $R(x) = (-2x^2 + 100x)$ CHF
- c) $C(x) = (20x^2 + 50 + 3e^{4x})$ CHF $R(x) = (200x - e^{-4x^2})$ CHF

R2.4 Determine all positions where the given function has ...

- i) ... a local maximum.
- ... a local minimum.
- ii) ... a point of inflection.
- a) $f(x) = 2x^3 - 9x^2 + 12x - 1$
- b) $f(x)$ as in R2.2 a)

R2.5 The total revenue function for a commodity or a service is given by

$$R(x) = (-0.01x^2 + 36x) \text{ CHF}$$

Determine the maximum revenue if production is limited to at most 1500 units.

R2.6 If the total cost function for a commodity or a service is

$$C(x) = (x^2 + 100) \text{ CHF}$$

producing or rendering how many units x will result in a minimum average cost?
Determine that minimum average cost.

R2.7 A firm can produce 1000 units per month only. The monthly total cost is given by

$$C(x) = (200x + 300) \text{ CHF}$$

where x is the number produced. The total revenue is given by

$$R(x) = \left(-\frac{1}{100}x^2 + 250x\right) \text{ CHF}$$

How many items should the firm produce for a maximum profit?
Determine that maximum profit.

R2.8 Determine the indefinite integrals below:

a) $\int (x^4 - 3x^3 - 6) \, dx$

b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right) \, dx$

R2.9 The equation of the third derivative f''' of a function f is given as follows:

$$f'''(x) = 3x + 1$$

Determine the equation of the function f such that $f''(0) = 0$, $f'(0) = 1$, $f(0) = 2$

R2.10 The marginal cost for producing a product or rendering a service is $C'(x) = (5x + 10)$ CHF, with a fixed cost of 800 CHF.

What will be the cost of producing or rendering 20 units?

R2.11 A certain firm's marginal cost $C'(x)$ and the derivative of the average revenue $\bar{R}'(x)$ are given as follows:

$$C'(x) = (6x + 60) \text{ CHF}$$

$$\bar{R}'(x) = -1 \text{ CHF}$$

If 10 items are produced or rendered, the total costs are 1000 CHF, and the revenue is 1700 CHF.

How many units will result in a maximum profit?
Determine that maximum profit.

R2.12 The supply function for a product or service is

$$p = f_s(x) = (4x + 4) \text{ CHF}$$

and the demand function is

$$p = f_d(x) = (-x^2 + 49) \text{ CHF}$$

Determine the equilibrium point and both the consumer's and the producer's surplus there.

R2.13 (see next page)

R2.13 The supply function for a product or a service is

$$p = f_s(x) = \left(ax^2 - \frac{6}{5}x + 2 \right) \text{ CHF}$$

and the demand function is

$$p = f_d(x) = (-bx^2 + 110) \text{ CHF}$$

with unknown parameters a and b . The equilibrium price is 10 CHF, and the producer's surplus is 73.33 CHF (rounded).

Determine the two unknown parameters a and b .

Hint:

- Use the unrounded value $\left(73 + \frac{1}{3} \right) \text{ CHF} = \frac{220}{3} \text{ CHF}$ for the producer's surplus.

Answers

- R2.1 a) true
 b) true
 c) false
 d) true
 e) true
 f) false
 g) true
 h) true

- R2.2 a) $f(x) = 4x^2(x^2 - 1)$ $f(-1) = 0$
 $f'(x) = 16x^3 - 8x$ $f'(-1) = -8$
 $f''(x) = 48x^2 - 8$ $f''(-1) = 40$
 b) $f(x) = (-3x^2 + 2x - 1) \cdot e^x$ $f(-2) = -17 \cdot e^{-2} = -2.300\dots$
 $f'(x) = (-3x^2 - 4x + 1) \cdot e^x$ $f'(-2) = -3 \cdot e^{-2} = -0.406\dots$
 $f''(x) = (-3x^2 - 10x - 3) \cdot e^x$ $f''(-2) = 5 \cdot e^{-2} = 0.676\dots$
 c) $f(x) = (x^2 + 2) \cdot e^{-3x}$ $f\left(-\frac{1}{3}\right) = \frac{19}{9}e = 5.738\dots$
 $f'(x) = (-3x^2 + 2x - 6) \cdot e^{-3x}$ $f'\left(-\frac{1}{3}\right) = -7e = -19.027\dots$
 $f''(x) = (9x^2 - 12x + 20) \cdot e^{-3x}$ $f''\left(-\frac{1}{3}\right) = 25e = 67.957\dots$

- R2.3 a) i) $C'(x) = 40$ CHF
 ii) $R'(x) = 60$ CHF
 iii) $P'(x) = R'(x) - C'(x) = 20$ CHF
 b) i) $C'(x) = (10x + 20)$ CHF
 ii) $R'(x) = (-4x + 100)$ CHF
 iii) $P'(x) = R'(x) - C'(x) = (-14x + 80)$ CHF
 c) i) $C'(x) = (40x + 12e^{4x})$ CHF
 ii) $R'(x) = (200 + 8x e^{-4x^2})$ CHF
 iii) $P'(x) = R'(x) - C'(x) = (-40x + 200 - 12e^{4x} + 8x e^{-4x^2})$ CHF

- R2.4 a) $f(x) = 2x^3 - 9x^2 + 12x - 1$
 $f'(x) = 6x^2 - 18x + 12$
 $f''(x) = 12x - 18$
 $f'''(x) = 12$
 i) $f'(x) = 0$ at $x_1 = 1$ and $x_2 = 2$
 $f''(x_1) = -6 < 0$ $\Rightarrow$ local maximum at $x_1 = 1$
 $f''(x_2) = 6 > 0$ $\Rightarrow$ local minimum at $x_2 = 2$
 ii) $f''(x) = 0$ at $x_3 = \frac{3}{2}$
 $f'''(x_3) = 12 \neq 0$ $\Rightarrow$ point of inflection at $x_3 = \frac{3}{2}$
 b) (see next page)

b) $f(x) = 4x^2(x^2 - 1) = 4x^4 - 4x^2$
 $f'(x) = 16x^3 - 8x$
 $f''(x) = 48x^2 - 8$
 $f'''(x) = 96x$

i) $f'(x) = 0$ at $x_1 = 0$, $x_2 = \frac{1}{\sqrt{2}}$, and $x_3 = -\frac{1}{\sqrt{2}}$
 $f''(x_1) = -8 < 0 \Rightarrow$ local maximum at $x_1 = 0$
 $f''(x_2) = 16 > 0 \Rightarrow$ local minimum at $x_2 = \frac{1}{\sqrt{2}}$
 $f''(x_3) = 16 > 0 \Rightarrow$ local minimum at $x_3 = -\frac{1}{\sqrt{2}}$

ii) $f''(x) = 0$ at $x_4 = \frac{1}{\sqrt{6}}$ and $x_5 = -\frac{1}{\sqrt{6}}$
 $f'''(x_4) = \frac{96}{\sqrt{6}} \neq 0 \Rightarrow$ point of inflection at $x_4 = \frac{1}{\sqrt{6}}$
 $f'''(x_5) = -\frac{96}{\sqrt{6}} \neq 0 \Rightarrow$ point of inflection at $x_5 = -\frac{1}{\sqrt{6}}$

R2.5 **Local** maximum at $x = 1800$ lies outside the possible interval $0 \leq x \leq 1500$.
 $R(1500) = 31'500 \text{ CHF} > R(0) = 0 \text{ CHF}$
 $\Rightarrow R = 31'500 \text{ CHF}$ is the **global** maximum revenue at $x = 1500$.

R2.6 $\bar{C}(x) = \frac{C(x)}{x} = \left(x + \frac{100}{x}\right) \text{ CHF}$
 $\bar{C}(x)$ has a **local** minimum at $x_1 = 10$.
 $\bar{C}(10) = 20 \text{ CHF}$
 $\bar{C}(x) > \bar{C}(x_1)$ if $x \neq x_1$ as there is no local maximum.
 $\Rightarrow \bar{C} = 20 \text{ CHF}$ is the **global** minimum average cost at $x = 10$.

R2.7 $P(x) = R(x) - C(x) = \left(-\frac{1}{100}x^2 + 50x - 300\right) \text{ CHF}$
 $P(x)$ has a **local** maximum at $x_1 = 2500$. This is outside the possible interval $0 \leq x \leq 1000$.
 $P(1000) = 39'700 \text{ CHF} > P(0) = -300 \text{ CHF}$
 $\Rightarrow P = 39'700 \text{ CHF}$ is the **global** maximum profit at the endpoint $x = 1000$.

R2.8 a) $\int (x^4 - 3x^3 - 6) \, dx = \frac{1}{5}x^5 - \frac{3}{4}x^4 - 6x + C$
b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right) \, dx = \frac{1}{14}x^7 + \frac{2}{9x^3} + C$

R2.9 $f(x) = \frac{1}{8}x^4 + \frac{1}{6}x^3 + x + 2$

R2.10 $C(20) = 2000 \text{ CHF}$

Hint:

- First, determine the cost function $C(x) \Rightarrow C(x) = \left(\frac{5}{2}x^2 + 10x + 800\right) \text{ CHF}$

R2.11 (see next page)

R2.11 $P = 800$ CHF is the global maximum profit at $x = 15$ units.

Hints:

- Determine the cost function $C(x) \Rightarrow C(x) = (3x^2 + 60x + 100)$ CHF
- Determine the average revenue function $\bar{R}(x) \Rightarrow \bar{R}(x) = (-x + C)$ CHF
- Determine the revenue function $R(x) \Rightarrow R(x) = (-x^2 + 180x)$ CHF
- Determine the profit function $P(x) \Rightarrow P(x) = (-4x^2 + 120x - 100)$ CHF
- The profit function $P(x)$ is a quadratic function.
- Think of the graph of the profit function when determining the global maximum.

R2.12 Equilibrium quantity

$$x = 5$$

Equilibrium price

$$p = 24 \text{ CHF}$$

Consumer's surplus

$$CS = 83.33 \text{ CHF (rounded)}$$

Producer's surplus

$$PS = 50 \text{ CHF}$$

R2.13 $a = \frac{1}{5}$
 $b = 1$