

Exercises 13 Applications of differential calculus Local/global maxima/minima, points of inflection

Objectives

- be able to determine the local maxima and minima of a function.
- be able to determine the points of inflection of a function.
- be able to determine the global maximum and the global minimum of a cost, revenue, and profit function.
- be able to determine the global minimum of an average cost, average revenue, and average profit function.

Problems

13.1 Determine all positions where the given function has ...

- i) ... a local maximum.
 ... a local minimum.
- ii) ... a point of inflection.
- a) $f(x) = x^2 - 4$
- b) $f(x) = -8x^3 + 12x^2 + 18x$
- c) $s(t) = t^4 - 8t^2 + 16$
- d) $f(x) = x e^{-x}$
- e) * $f(x) = (1 - e^{-2x})^2$
- f) * $V(r) = -D \left(\frac{2a}{r} - \frac{a^2}{r^2} \right) \quad (D > 0, a > 0)$

13.2 If the total profit for a commodity is

$$P(x) = (2000x + 20x^2 - x^3) \text{ CHF}$$

where x is the number of items sold, determine the level of sales, x , that maximises profit, and find the maximum profit.

Hints:

- First, find the local maxima.
- Then, check if one of the local maxima is the global maximum.

13.3 If the total cost for a service concerning a tourism event is given by

$$C(x) = \left(\frac{1}{4}x^2 + 4x + 100 \right) \cdot 100 \text{ CHF}$$

where x represents the extent of the service, what value of x will result in a minimum average cost? Determine the minimum average cost.

13.4 Suppose that the production capacity for a certain commodity cannot exceed 30. If the total profit for this company is

$$P(x) = (4x^3 - 210x^2 + 3600x) \text{ CHF}$$

where x is the number of units sold, determine the number of items that will maximise profit.

13.5 (see next page)

- 13.5 Suppose the annual profit for a store is given by

$$P(x) = (-0.1x^3 + 3x^2) \cdot 1000 \text{ CHF}$$

where x is the number of years past 2010. If this model is accurate, determine the point of inflection for the profit.

- 13.6 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

- a) If f has a local maximum at x_0 it can be concluded that ...

- ☐ ... $f(x_0) > f(x)$ for any $x \neq x_0$
☐ ... $f(x_0) > f(x)$ for any $x > x_0$
☐ ... $f(x_0) > f(x)$ for any $x < x_0$
☐ ... $f(x_0) > f(x)$ for all x which are in a certain neighbourhood of x_0

- b) If $f(x_0) < 0$, $f'(x_0) = 0$, and $f''(x_0) \neq 0$, it can be concluded that f has ...

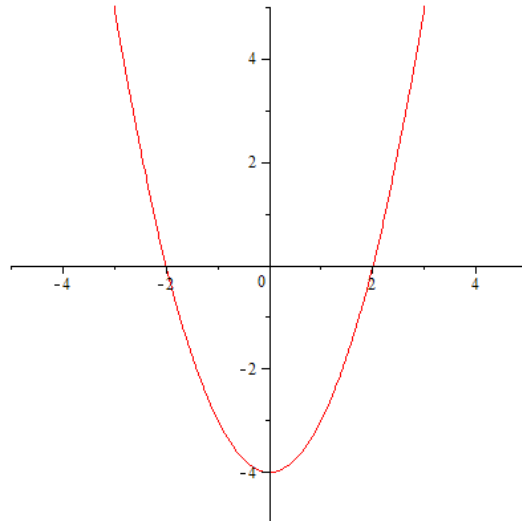
- ☐ ... no local minimum at x_0
☐ ... no local maximum at x_0
☐ ... no point of inflection at x_0
☐ ... a point of inflection at x_0

- c) The global maximum of a function ...

- ☐ ... is always a local maximum.
☐ ... can be a local minimum.
☐ ... can be a local maximum.
☐ ... always exists.

Answers

13.1 a) $f(x) = x^2 - 4$



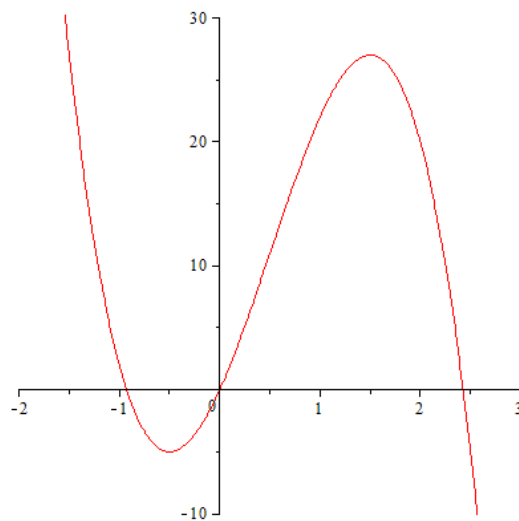
$$f'(x) = 2x$$

$$f''(x) = 2$$

$$f'''(x) = 0$$

- i) $f'(x) = 0$ at $x_1 = 0$
 $f''(x_1) = 2 > 0 \quad \Rightarrow \quad \begin{array}{l} \text{local minimum at } x_1 = 0 \\ \text{no local maximum} \end{array}$
- ii) $f''(x) = 2 \neq 0$ for all $x \quad \Rightarrow \quad \text{no point of inflection}$

b) $f(x) = -8x^3 + 12x^2 + 18x$



$$f'(x) = -24x^2 + 24x + 18$$

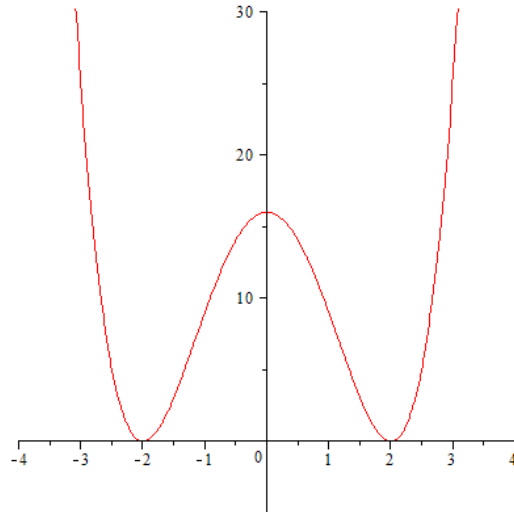
$$f''(x) = -48x + 24$$

$$f'''(x) = -48$$

- i) $f'(x) = 0$ at $x_1 = -\frac{1}{2}$ and $x_2 = \frac{3}{2}$
 $f''(x_1) = 48 > 0 \quad \Rightarrow \quad \text{local minimum at } x_1 = -\frac{1}{2}$
 $f''(x_2) = -48 < 0 \quad \Rightarrow \quad \text{local maximum at } x_2 = \frac{3}{2}$

$$\begin{aligned} \text{ii)} \quad f''(x) &= 0 \text{ at } x_3 = \frac{1}{2} \\ f'''(x_3) &= -48 \neq 0 \end{aligned} \quad \Rightarrow \quad \text{point of inflection at } x_3 = \frac{1}{2}$$

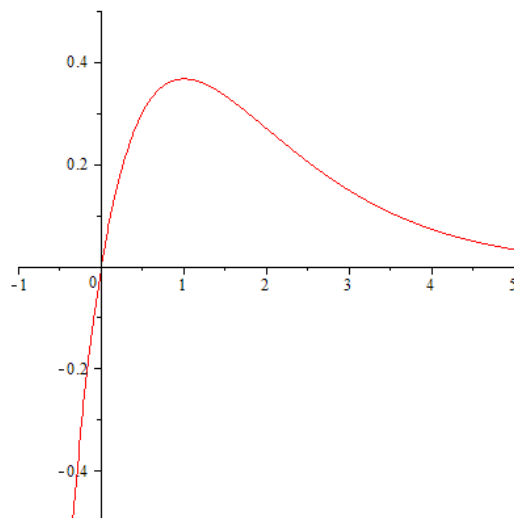
c) $s(t) = t^4 - 8t^2 + 16$



$$\begin{aligned} s'(t) &= 4t^3 - 16t \\ s''(t) &= 12t^2 - 16 \\ s'''(t) &= 24t \end{aligned}$$

$$\begin{aligned} \text{i)} \quad s'(t) &= 0 \text{ at } t_1 = 0, t_2 = -2, \text{ and } t_3 = 2 \\ s''(t_1) &= -16 < 0 & \Rightarrow & \text{local maximum at } t_1 = 0 \\ s''(t_2) &= 32 > 0 & \Rightarrow & \text{local minimum at } t_2 = -2 \\ s''(t_3) &= 32 > 0 & \Rightarrow & \text{local minimum at } t_3 = 2 \\ \text{ii)} \quad s''(t) &= 0 \text{ at } t_4 = -\frac{2}{\sqrt{3}} \text{ and } t_5 = \frac{2}{\sqrt{3}} \\ s'''(t_4) &= -\frac{48}{\sqrt{3}} \neq 0 & \Rightarrow & \text{point of inflection at } t_4 = -\frac{2}{\sqrt{3}} \\ s'''(t_5) &= \frac{48}{\sqrt{3}} \neq 0 & \Rightarrow & \text{point of inflection at } t_5 = \frac{2}{\sqrt{3}} \end{aligned}$$

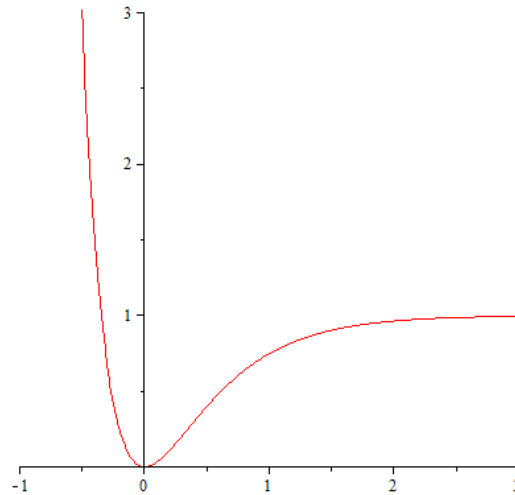
d) $f(x) = x e^{-x}$



$$\begin{aligned} f'(x) &= e^{-x} - x e^{-x} = (1 - x) e^{-x} \\ f''(x) &= -e^{-x} - (1 - x) e^{-x} = (x - 2) e^{-x} \\ f'''(x) &= e^{-x} - (x - 2) e^{-x} = (3 - x) e^{-x} \end{aligned}$$

- i) $f'(x) = 0$ at $x_1 = 1$
 $f''(x_1) = -\frac{1}{e} < 0 \Rightarrow$ local maximum at $x_1 = 1$
no local minimum
- ii) $f''(x) = 0$ at $x_2 = 2$
 $f'''(x_2) = \frac{1}{e^2} \neq 0 \Rightarrow$ point of inflection at $x_2 = 2$

e) * $f(x) = (1 - e^{-2x})^2 = 1 - 2e^{-2x} + e^{-4x}$



$$f'(x) = 4e^{-2x} - 4e^{-4x} = 4e^{-2x}(1 - e^{-2x})$$

$$f''(x) = -8e^{-2x} + 16e^{-4x} = 8e^{-2x}(2e^{-2x} - 1)$$

$$f'''(x) = 16e^{-2x} - 64e^{-4x} = 16e^{-2x}(1 - 4e^{-2x})$$

- i) $f'(x) = 0$ at $x_1 = 0$
 $f''(x_1) = 8 > 0 \Rightarrow$ local minimum at $x_1 = 0$
no local maximum
- ii) $f''(x) = 0$ at $x_2 = \frac{\ln(2)}{2} = 0.34\dots$
 $f'''(x_2) = -8 \neq 0 \Rightarrow$ point of inflection at $x_2 = 0.34\dots$

f) * $V'(r) = -D\left(-\frac{2a}{r^2} + \frac{2a^2}{r^3}\right) = \frac{2aD}{r^2}\left(1 - \frac{a}{r}\right)$
 $V''(r) = -D\left(\frac{4a}{r^3} - \frac{6a^2}{r^4}\right) = \frac{2aD}{r^3}\left(\frac{3a}{r} - 2\right)$
 $V'''(r) = -D\left(-\frac{12a}{r^4} + \frac{24a^2}{r^5}\right) = \frac{12aD}{r^4}\left(1 - \frac{2a}{r}\right)$

- i) $V'(r) = 0$ at $r_1 = a$
 $V''(r_1) = \frac{2D}{a^2} > 0 \Rightarrow$ local minimum at $r_1 = a$
no local maximum
- ii) $V''(r) = 0$ at $r_2 = \frac{3a}{2}$
 $V'''(r_2) = -\frac{64D}{81a^3} \neq 0 \Rightarrow$ point of inflection at $r_2 = \frac{3a}{2}$

13.2 (Sole) **local** maximum at $x_1 = \frac{100}{3} \rightarrow 33$ or 34

$P(33) = 51'843$ CHF

$P(34) = 51'816$ CHF

$P(x) < P(x_1)$ if $x \neq x_1$ as there is no local minimum

$\Rightarrow P = 51'843$ CHF is the **global** maximum profit at $x = 33$.

13.3 $\bar{C}(x) = \frac{C(x)}{x} = \left(\frac{1}{4}x + 4 + \frac{100}{x}\right) \cdot 100 \text{ CHF}$

$\bar{C}(x)$ has a (sole) **local** minimum at $x_1 = 20$.

$\bar{C}(20) = 1400 \text{ CHF}$

$\bar{C}(x) > \bar{C}(x_1)$ if $x \neq x_1$ as there is no local maximum.

$\Rightarrow \bar{C} = 1400 \text{ CHF}$ is the **global** minimum average cost at $x = 20$.

13.4 $P(x)$ has a **local** maximum at $x_1 = 15$ and a **local** minimum at $x_2 = 20$.

$P(x_1) = 20'250 \text{ CHF}$

$P(x) < P(x_1)$ if $x < x_1$ as there is no local minimum on the interval $x < x_1$.

$P(30) = 27'000 \text{ CHF} > 20'250 \text{ CHF} (!)$

$\Rightarrow P = 27'000 \text{ CHF}$ is the **global** maximum profit at the endpoint $x = 30$.

13.5 $P(x)$ has a point of inflection at $x_1 = 10$.

$P(10) = 200 \cdot 1000 \text{ CHF} = 200'000 \text{ CHF}$

$\Rightarrow$ point of inflection (10 | 200'000 CHF), i.e. when $x = 10$ (in the year 2020) and $P = 200'000 \text{ CHF}$

13.6 a) 4th statement

b) 3rd statement

c) 3rd statement