

Exercises 8 Exponential function and equations Compound interest, exponential function

Objectives

- be able to perform compound interest calculations.
- be able to graph an exponential function out of its equation.
- be able to determine the equation of an exponential function out of the coordinates of two points of the graph.
- be able to treat applied tasks by means of an exponential function.

Problems

Compound interest

- 8.1 Compound interest at an interest rate r is paid on an initial capital C_0 .
- Assume an initial capital $C_0 = 1000.00$ CHF, and an interest rate $r = 2\%$. Determine the capital after one, two, three, four, and five compounding periods.
 - Try to develop a formula which allows you to calculate the capital C_n after n compounding periods for any values of C_0 , r , and n .
 - Solve the formula that you have developed in b) for C_0 and r .
- 8.2 What is the future capital if 8000 CHF are invested for 10 years at an annual interest rate of 12%, compounded annually?
- 8.3 What present value amounts to 10'000 CHF if it is invested for 10 years at an annual interest rate of 6%, compounded annually?
- 8.4 At what annual interest rate, compounded annually, would 10'000 CHF have to be invested to amount to 14'000 CHF in 7 years?
- 8.5 Ms Smith wants to invest 150'000 CHF for five years. Bank A offers an annual interest rate of 6.5%, compounded annually. Bank B offers to pay 200'000 CHF after five years. Which bank makes the better offer?
- 8.6 Mary Stahley invested 2500 CHF in a 36-month certificate of deposit (CD) that earned 8.5% annual **simple** interest. When the CD matured, she invested the full amount in a mutual fund that had an annual growth equivalent to 18%, **compounded** annually. How much was the mutual fund worth 9 years later?
- 8.7 A capital is invested for 4 years at 4% and for 3 more years at 6%, compounded annually. Eventually, the capital amounts to 72'000 CHF.
- Determine the initial capital.
 - What is the average interest rate with respect to the whole period of time?
- 8.8 An unknown initial capital is invested at an unknown annual interest rate, compounded annually. After 2 years, the capital amounts to 5'891.74 CHF (rounded), and after another 5 years the capital is 6'997.54 CHF (rounded). Determine both initial capital (rounded to 100 CHF) and annual interest rate (rounded to 0.1%).

- 8.9 A capital pays interest, compounded annually.
Determine the annual interest rate such that the capital doubles in 20 years.
- 8.10 What is the future value if 3200 CHF is invested for 5 years at a nominal annual interest rate of 8%, compounded quarterly?
- 8.11 What amount of money do parents need to deposit in an account earning 10% (nominal annual interest rate), compounded monthly, so that it will grow to 40'000 CHF for their son's college tuition in 18 years?
- 8.12 A certain capital is invested at a nominal annual interest rate of 6%. By how many percent does the capital grow in one year if interest is compounded ...
- a) ... annually?
 - b) ... semiannually?
 - c) ... quarterly?
 - d) ... monthly?
 - e) ... daily (1 year = 360 days)?

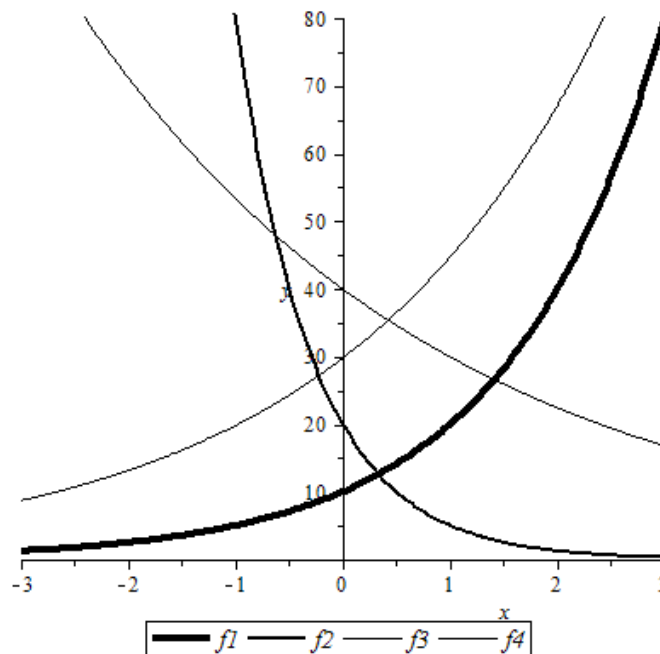
Exponential function

- 8.13 Look at the following exponential function:

$$\begin{aligned} f: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f(x) = 2^x \end{aligned}$$

- a) Establish a table of values of f for the interval $-3 \leq x \leq 3$.
 - b) Draw the graph of f in the interval $-3 \leq x \leq 3$ into a Cartesian coordinate system.
- 8.14 Graph the following exponential functions into one coordinate system:
- $$\begin{aligned} f_1: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f_1(x) = 2^x \end{aligned}$$
- $$\begin{aligned} f_2: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f_2(x) = 0.2^x \end{aligned}$$
- $$\begin{aligned} f_3: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f_3(x) = 3 \cdot 0.5^x \end{aligned}$$
- $$\begin{aligned} f_4: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f_4(x) = -2 \cdot 3^x \end{aligned}$$
- 8.15 (see next page)

8.15 Look at the graphs of the exponential functions f_1 , f_2 , f_3 , and f_4 :



Determine the equations of the four functions, i.e. $y = f(x) = \dots$

8.16 The graph of an exponential function contains the points P and Q. Determine the equation of the exponential function.

- | | | |
|----|-----------|-----------------------|
| a) | P(1 12) | Q(3 192) |
| b) | P(0 1.02) | Q(1 1.0302) |
| c) | P(5 16) | Q(9 $\frac{1}{16}$) |

8.17 A flat that 20 years ago was worth 160'000 CHF has increased in value by 4% each year due to the market situation. What is the flat worth today?

8.18 A machine is valued at 10'000 CHF. The depreciation at the end of each year is 20% of its value at the beginning of the year. Find its value at the end of 4 years.

8.19 The size of a certain bacteria culture grows exponentially. At 8 a.m. and 11 a.m. the number of bacteria was 2'300 and 18'400, respectively. Determine the number of bacteria at 1.30 p.m.

8.20 Decide which statements are true or false. Put a mark into the corresponding box. In each problem a) to c), exactly one statement is true.

a) In a compound interest scheme ...

- | | |
|--------------------------|--|
| ☐ | ... the graph that represents the growth of the capital is a parabola. |
| ☐ | ... the interest paid at the end of each period only depends on the interest rate. |
| ☐ | ... the interest rate depends on the capital of the previous period. |
| ☐ | ... the capital grows exponentially. |

b) (see next page)

b) The graph of an exponential function ...

☐

... is a parabola.

☐

... is a hyperbola.

☐

... never intersects the y-axis.

☐

... never touches the x-axis.

c) If a quantity grows exponentially in time ...

☐

... the growth factor itself grows.

☐

... the growth factor depends on the initial value.

☐

... the quantity doubles in one year if the annual growth factor is 100%.

☐

... the quantity doubles in constant time intervals.

Answers

- 8.1 a) $C_0 = 1000.00 \text{ CHF}$ $C_1 = 1020.00 \text{ CHF}$ $C_2 = 1040.40 \text{ CHF}$
 $C_3 = 1061.21 \text{ CHF (rounded)}$ $C_4 = 1082.43 \text{ CHF (rounded)}$ $C_5 = 1104.08 \text{ CHF (rounded)}$
 b) $C_n = C_0 (1 + r)^n$
 c) see [formulary](#)
- 8.2 $C_n = C_0 (1 + r)^n$ where $C_0 = 8000 \text{ CHF}$, $r = 12\%$, $n = 10$
 $\Rightarrow C_{10} = 24'846.79 \text{ CHF (rounded)}$
- 8.3 $C_0 = \frac{C_n}{(1 + r)^n}$ where $C_n = 10'000 \text{ CHF}$, $r = 6\%$, $n = 10$
 $\Rightarrow C_0 = 5'583.95 \text{ CHF (rounded)}$
- 8.4 $r = \sqrt[n]{\frac{C_n}{C_0}} - 1$ where $C_0 = 10'000 \text{ CHF}$, $C_n = 14'000 \text{ CHF}$, $n = 7$
 $\Rightarrow r = 4.9\% \text{ (rounded)}$
- 8.5 Bank A: $C_5 = 205'513.00 \text{ CHF (rounded)}$
 Bank B: $C_5 = 200'000.00 \text{ CHF}$
- 8.6 13'916.24 CHF
 2 periods: 3 years of simple interest, 9 years of compound interest
 - 3 years of simple interest:
 $C_n = C_0(1 + nr)$ where $C_0 = 2500 \text{ CHF}$, $r = 8.5\%$, $n = 3$
 $\Rightarrow C_3 = 3137.50 \text{ CHF}$
 - 9 years of compound interest:
 $C_n = C_0 (1 + r)^n$ where $C_0 = \dots (= C_3 \text{ after first 3 years})$, $r = 18\%$, $n = 9$
 $\Rightarrow C_9 = 13'916.24 \text{ CHF (rounded)}$
- 8.7 a) $C_0 = 51'675 \text{ CHF (rounded)}$
 Hints:
 - First, look at the second period (3 years, starting after 4 years from now), and calculate the capital at the beginning of this second period.
 - Then, calculate the initial capital.
 b) $i = 4.85\% \text{ (gerundet)}$
 Hints:
 - There are two possible solution processes:
 - **Process 1**
 The average interest rate r must be such that:
 $C_n = C_0 (1 + r)^n = C_0 (1 + r_1)^{n_1} (1 + r_2)^{n_2}$ and $n_1 + n_2 = n$
 where $r_1 =$ interest rate in the first n_1 periods, $r_2 =$ interest rate in the remaining n_2 periods
 - **Process 2**
 The average interest rate r must be such that:
 $C_n = C_0 (1 + r)^n$
 where $C_0 =$ initial capital, $C_n =$ capital after the whole n periods

8.8 $r = 3.5\%$, $C_0 = 5'500.00$ CHF

Hints:

- First, look at the second period of 5 years, where $C_0 = 5'891.74$ CHF and $C_5 = 6'997.54$ CHF.
- The 5'891.74 CHF can be considered as the capital C_2 at the end of the first 2 years if C_0 is the initial capital at the beginning of the whole 7 years.

8.9 $r = \sqrt[n]{\frac{C_n}{C_0}} - 1$ where $n = 20$
 $C_n = 2 \cdot C_0$

 $\Rightarrow r = \sqrt[20]{2} - 1 = 3.5\%$ (rounded)

8.10 $C_n = C_0 (1 + r)^n$ where $C_0 = 3200$ CHF, $r = \frac{8\%}{4} = 2\%$, $n = 5 \cdot 4 = 20$
 $\Rightarrow C_{20} = 4755.03$ CHF (rounded)

8.11 $C_0 = \frac{C_n}{(1 + r)^n}$ where $C_n = 40'000$ CHF, $r = \frac{10\%}{12}$, $n = 18 \cdot 12 = 216$
 $\Rightarrow C_0 = 6661.46$ CHF (rounded)

8.12 Capital after 1 year
 $C_n = C_0 (1 + r)^n$ (n = number of compounding periods per year, r = interest rate per compounding period)
 $C_n = C_0 (1 + x)$ (x = asked percentage)
 $\Rightarrow x = (1 + r)^n - 1$

- | | | |
|----|--------------------------------|-------------------------|
| a) | $n = 1, r = 6\%$ | $x = 6\%$ |
| b) | $n = 2, r = \frac{6\%}{2}$ | $x = 6.09\%$ |
| c) | $n = 4, r = \frac{6\%}{4}$ | $x = 6.136\%$ (rounded) |
| d) | $n = 12, r = \frac{6\%}{12}$ | $x = 6.168\%$ (rounded) |
| e) | $n = 360, r = \frac{6\%}{360}$ | $x = 6.183\%$ (rounded) |

8.13 ...

8.14 ...

8.15 $y = f_1(x) = 10 \cdot 2^x$ ($c = 10, a = 2$)
 $y = f_2(x) = 20 \cdot 0.25^x$ ($c = 20, a = 0.25$)
 $y = f_3(x) = 30 \cdot 1.5^x$ ($c = 30, a = 1.5$)
 $y = f_4(x) = 40 \cdot 0.75^x$ ($c = 40, a = 0.75$)

8.16 (see next page)

8.16 a) $y = f(x) = 3 \cdot 4^x$

Hints:

- The equation of an exponential function is $y = f(x) = c \cdot a^x$
- If $P(1|12)$ and $Q(3|192)$ are points of the graph of the exponential function, their coordinates must fulfil the equation of the exponential function, i.e. $12 = f(1) = c \cdot a^1$ and $192 = f(3) = c \cdot a^3$
- Solve the two equations for c and a .

b) $y = f(x) = 1.02 \cdot 1.01^x$

c) $y = f(x) = 16'384 \cdot 0.25^x$

8.17 350'580 CHF (rounded)

Hint:

- The relation between time t (t = number of years elapsed since 20 years ago) and the value V of the house is an exponential function:

$$V = f(t) = V_0 \cdot a^t$$

where V = value at time t , V_0 = initial value (at $t = 0$) = 160'000 CHF, a = growth factor = $1 + 4\% = 1.04$

8.18 4'096 CHF

8.19 104'086 (rounded)

8.20 a) 4th statement

b) 4th statement

c) 4th statement