

Exercises 12 **Differentiation rules** **Coefficient, sum, product, exponential function, higher-order derivatives**

Objectives

- be able to apply the coefficient, sum, and product rules to determine the derivative of a function.
- be able to determine a higher-order derivative of a function.

Problems

12.1 Determine the derivative by applying the **coefficient rule**:

- | | | |
|-------------------------------------|------------------------------|-------------------------|
| a) $f(x) = 3x^5$ | b) $f(x) = -4x^3$ | c) $f(x) = -x^{10}$ |
| d) $f(x) = a \cdot x^3$ | e) $f(x) = n \cdot x^{n-1}$ | f) $f(x) = 9 \cdot 3^x$ |
| g) $s(t) = \frac{1}{2} g \cdot t^2$ | h) $S(T) = \alpha \cdot T^4$ | i) $C(x) = (-3x)^3$ |

12.2 Determine the derivative by applying the **sum rule**:

- | | | |
|---|--|---|
| a) $f(x) = x^5 + x^6$ | b) $f(x) = x^{10} - x^9$ | c) $f(x) = 1 + x + 3x^3$ |
| d) $f(x) = \frac{1}{4}x^4 + 3x^2 - 2$ | e) $f(x) = 3x^2(x - 2)$ | f) $f(x) = -3x^8 + x^5 - 3x + 99$ |
| g) $f(x) = ax^2 + bx + c$ | h) $f(x) = 3(a^2 - 2ax + x^2)$ | i) $f(x) = \frac{x^3}{3} - \frac{3}{x^3}$ |
| j) $s(t) = s_0 + v_0t + \frac{1}{2}g \cdot t^2$ | k) $V(r) = -\frac{a}{r} + \frac{b}{r^2}$ | l) $C(n) = C_0(1 + nr)$ |

Hint:

- In some problems, the coefficient rule is needed, too.

12.3 Determine the derivative by applying the **product rule**:

- | | |
|-------------------------------------|--|
| a) $f(x) = x \cdot e^x$ | b) $f(x) = x^3 \cdot 3^x$ |
| c) $f(x) = -2x^5(x - 1)$ | d) $f(x) = (2x - 1) \cdot e^x$ |
| e) $f(x) = (2x - 1)(-3x^2 - x + 1)$ | f) $V(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3} \right)$ |

Hint:

- In some problems, the coefficient and/or the sum rule(s) is/are needed, too.

12.4 Determine the derivative of the exponential functions below:

- | | |
|----------------------|--------------------------|
| a) $f(x) = e^{4x}$ | b) $f(x) = e^{-x}$ |
| c) $f(x) = e^{-x^2}$ | d) $f(x) = e^{x^2-2x+5}$ |

12.5 Determine the derivative of the functions below. Apply the appropriate differentiation rule(s). Simplify and factorise the derivative as far as possible:

- | | |
|---|------------------------------|
| a) $f(x) = (x - 2) e^{2x}$ | b) $f(x) = (2 - x^2) e^{-x}$ |
| c) $f(x) = (3x^3 - 2x^2 + x - 1) e^{-2x}$ | d) $P(v) = av^2 e^{-bv^2}$ |

12.6 (see next page)

12.6 Determine the derivatives (rates of change) below:

- a) $f'(2)$ with function f in 12.1 b)
- b) $s'(4)$ with function s in 12.1 g)
- c) $f'(-1)$ with function f in 12.2 g)
- d) $P'(1)$ with function P in 12.5 d)

12.7 Determine the second and third derivatives of the functions below. Simplify and factorise the higher-order derivatives as far as possible:

- a) Function f in 12.1 a)
- b) Function f in 12.2 g)
- c) Function f in 12.3 a)
- d) Function f in 12.4 c)

Hint:

- You have already determined the first derivatives of the corresponding functions.

12.8 Determine the indicated higher-order derivatives:

- a) $f''(-1)$ with function f in 12.1 a)

Hint:

- You have already determined $f''(x)$ in 12.7 a).

- b) $f'''(2)$ with function f in 12.4 c)

Hint:

- You have already determined $f'''(x)$ in 12.7 d).

12.9 Decide which statements are true or false. Put a mark into the corresponding box. In each problem a) to c), exactly one statement is true.

- a) The third derivative of a function is a ...

- ... constant function if the second derivative is a quadratic function.
- ... quadratic function if the second derivative is a linear function.
- ... linear function if the first derivative is a quadratic function.
- ... constant function if the first derivative is a quadratic function.

- b) The derivative of a ...

- ... product is the product of the derivatives of the single factors.
- ... product is the sum of the derivatives of the single factors.
- ... sum is the sum of the derivatives of the single addends.
- ... constant is the constant itself.

- c) If $f(x) = c \cdot g(x) \cdot h(x)$ then $f'(x) = \dots$

- ... 0
- ... $c \cdot g'(x) \cdot h'(x)$
- ... $c \cdot g(x) \cdot h'(x) + c \cdot g'(x) \cdot h(x)$
- ... $c \cdot g'(x) \cdot h'(x) + c \cdot g(x) \cdot h(x)$

Answers

- 12.1 a) $f'(x) = 3 \cdot 5x^4 = 15x^4$
 b) $f'(x) = (-4) 3x^2 = -12x^2$
 c) $f'(x) = (-1) 10x^9 = -10x^9$
 d) $f'(x) = a \cdot 3x^2 = 3ax^2$

Hint:

- a is a constant.

- e) $f'(x) = n(n-1)x^{n-2}$
 f) $f'(x) = 9 \cdot 3^x \cdot \ln(3)$
 g) $s'(t) = \frac{g}{2} 2t = gt$

Hints:

- The name of the function is s, and the variable is t.
 - g is a constant.

- h) $S'(T) = \alpha \cdot 4T^3 = 4\alpha T^3$
 i) $C'(x) = -81x^2$

- | | | | | | |
|---------|-----------------------|----|--|----|-------------------------------|
| 12.2 a) | $f'(x) = 5x^4 + 6x^5$ | b) | $f'(x) = 10x^9 - 9x^8$ | c) | $f'(x) = 1 + 9x^2$ |
| d) | $f'(x) = x^3 + 6x$ | e) | $f'(x) = 9x^2 - 12x$ | f) | $f'(x) = -24x^7 + 5x^4 - 3$ |
| g) | $f'(x) = 2ax + b$ | h) | $f'(x) = -6a + 6x$ | i) | $f'(x) = x^2 + \frac{9}{x^4}$ |
| j) | $s'(t) = v_0 + gt$ | k) | $V'(r) = \frac{a}{r^2} - \frac{2b}{r^3}$ | l) | $C'(n) = C_0 r$ |

- 12.3 a) $f'(x) = e^x + x \cdot e^x$
 b) $f'(x) = 3x^2 \cdot 3^x + x^3 \cdot 3^x \cdot \ln(3)$
 c) $f'(x) = -2(5x^4(x-1) + x^5)$
 d) $f'(x) = 2 \cdot e^x + (2x-1) \cdot e^x$
 e) $f'(x) = 2(-3x^2 - x + 1) + (2x-1)(-6x-1)$
 f) $V'(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3} \right) + e^r \left(2a \cdot r + \frac{3b}{r^4} \right)$

Hints:

- V is the name of the function, and r is the variable.
 - a and b are constants.

- | | | | |
|---------|------------------------------|----|---------------------------------|
| 12.4 a) | $f'(x) = 4 e^{4x}$ | b) | $f'(x) = (-1) e^{-x} = -e^{-x}$ |
| c) | $f'(x) = -2x \cdot e^{-x^2}$ | d) | $f'(x) = (2x-2) e^{x^2-2x+5}$ |

- 12.5 a) $f'(x) = e^{2x} + (x-2) 2 e^{2x} = (2x-3) e^{2x}$
 b) $f'(x) = -2x e^{-x} + (2-x^2)(-1) e^{-x} = (x^2-2x-2) e^{-x}$
 c) $f'(x) = (9x^2-4x+1) e^{-2x} + (3x^3-2x^2+x-1)(-2) e^{-2x} = (-6x^3+13x^2-6x+3) e^{-2x}$
 d) $P'(v) = a \left(2v e^{-bv^2} + v^2(-2bv) e^{-bv^2} \right) = 2av(1-bv^2) e^{-bv^2}$

12.6 (see next page)

- 12.6 a) 12.1 b)
 $f'(2) = -48$ b) 12.1 g)
 $s'(4) = 4g$
- c) 12.2 g)
 $f'(-1) = -2a + b$ d) 12.5 d)
 $P'(1) = 2a(1 - b)e^{-b}$
- 12.7 a) 12.1 a)
 $f''(x) = 15 \cdot 4x^3 = 60x^3$
 $f'''(x) = 60 \cdot 3x^2 = 180x^2$
- b) 12.2 g)
 $f''(x) = 2a \cdot 1 = 2a$
 $f'''(x) = 0$
- c) 12.3 a)
 $f''(x) = e^x + (e^x + x \cdot e^x) = (x + 2) e^x$
 $f'''(x) = e^x + (x + 2) e^x = (x + 3) e^x$
- d) 12.4 c)
 $f''(x) = -2(e^{-x^2} + x(-2x)e^{-x^2}) = 2(2x^2 - 1)e^{-x^2}$
 $f'''(x) = 2(4xe^{-x^2} + (2x^2 - 1)(-2x)e^{-x^2}) = 4x(-2x^2 + 3)e^{-x^2}$
- 12.8 a) $f''(-1) = 60(-1)^3 = -60$
- b) $f'''(2) = 4 \cdot 2(-2 \cdot 2^2 + 3)e^{-2^2} = -\frac{40}{e^4}$
- 12.9 a) 4th statement
- b) 3rd statement
- c) 3rd statement