Exercises 16 Indefinite integral
Antiderivative, indefinite integral, coefficient/sum rule

Objectives

- be able to determine an antiderivative and the indefinite integral of a constant, basic power, and basic exponential function.
- be able to apply the coefficient and sum rules to determine the indefinite integral of a function.
- be able to determine the cost, revenue, and profit functions if the marginal cost, marginal revenue, and marginal profit functions are known.

Problems

16.1 Determine the indefinite integrals below:

a) $\int x^2\,dx$ b) $\int x^3\,dx$

c) $\int x^{-5}\,dx$ d) $\int \frac{1}{x^2}\,dx$

e) $\int \frac{1}{x^4}\,dx$ f) $\int 4\,dx$

g) $\int (-7)\,dx$ h) $\int e^x\,dx$

i) $\int e^{3x}\,dx$ j) $\int e^{-x}\,dx$

16.2 Determine the indefinite integral of the following functions f:

a) $f(x) = x^5$ b) $f(x) = 3x^2$

c) $f(x) = x^3 + 2x^2 - 5$ d) $f(x) = \frac{x^5}{2} - \frac{2}{3x^2}$

e) $f(x) = \frac{1}{2}x^3 - 2x^2 + 4x - 5$ f) $f(x) = x^{10} - \frac{1}{2}x^3 - x$

16.3 Determine the equations of those two antiderivatives F_1 and F_2 of f which fulfil the stated conditions.

a) $f(x) = 10x^2 + x$ $F_1(0) = 3$ $F_2(0) = -1$

b) $f(x) = x^3 + 3x + 1$ $F_1(2) = 5$ $F_2(4) = -8$

16.4 Suppose that we know the equation of the derivative f ' of a function f:

$$f'(x) = 3x^2 - 50x + 250$$

Determine the equation of the function f, if …

a) … $f(0) = 500$.

b) … $f(10) = 2500$.

16.5 Suppose that we know the equation of the second derivative f '' of a function f:

$$f''(x) = 2x - 1$$

Determine the equation of the function f such that $f'(2) = 4$ and $f(1) = -1$.

16.6 If the monthly marginal cost for a product is $C'(x) = (2x + 100)$ CHF, with fixed costs amounting to 200 CHF, determine the total cost function for a month.

16.7 If the marginal cost for a product is $C'(x) = (4x + 2)$ CHF, and the production of 10 units results in a total cost of 300 CHF, determine the total cost function.

16.8 If the marginal cost for a product is $C'(x) = (4x + 40)$ CHF, and the total cost of producing 25 units is 3000 CHF, what will be the total cost for 30 units?

16.9 A firm knows that its marginal cost for a product is $C'(x) = (3x + 20)$ CHF, that its marginal revenue is $R'(x) = (- 5x + 44)$ CHF, and that the cost of production and sale of 10 units is 370 CHF.

Determine the …

a) … profit function $P(x)$.

b) … number of units that results in a maximum profit

Hint:
- The revenue R is zero if no unit is sold. Thus, $R(0) = 0$ CHF.

16.10 Suppose that the marginal revenue $R'(x)$ and the derivative of the average cost $\overline{C}'(x)$ of a company are given as follows:

$$R'(x) = 400 \text{ CHF}$$

$$\overline{C}'(x) = \left(\frac{2}{15}x - 11 - \frac{10'000}{x^2}\right) \text{ CHF}$$

The production of 15 units results in a total cost of 16'750 CHF.

Determine the …

a) … profit function $P(x)$.

b) … number of units that results in a maximum profit.

c) … maximum profit.

16.11 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) An antiderivative of a function is a ...

- ☐ ... real number.
- ☐ ... function.
- ☐ ... set of functions.
- ☐ ... graph.

b) The indefinite integral of a function is a ...

- ☐ ... real number.
- ☐ ... function.
- ☐ ... set of functions.
- ☐ ... graph.

c) If $f = g'$ then ...

- ☐ ... f is an antiderivative of g.
- ☐ ... g is an antiderivative of f.
- ☐ ... f is the indefinite integral of g.
- ☐ ... g is the indefinite integral of f.

Answers

16.1 a) $\int x^2\,dx = \frac{1}{3}x^3 + C$ b) $\int x^3\,dx = \frac{1}{4}x^4 + C$

c) $\int x^{-5}\,dx = -\frac{1}{4x^4} + C$ d) $\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$

e) $\int \frac{1}{x^4}\,dx = -\frac{1}{3x^3} + C$ f) $\int 4\,dx = 4x + C$

g) $\int (-7)\,dx = -7x + C$ h) $\int e^x\,dx = e^x + C$

i) $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$ j) $\int e^{-x}\,dx = -e^{-x} + C$

16.2 a) $\int f(x)\,dx = \int x^5\,dx = \frac{1}{6}x^6 + C$

b) $\int f(x)\,dx = \int 3x^2\,dx = x^3 + C$

c) $\int f(x)\,dx = \int (x^3 + 2x^2 - 5)\,dx = \frac{1}{4}x^4 + \frac{2}{3}x^3 - 5x + C$

d) $\int f(x)\,dx = \int \left(\frac{1}{2}x^5 - \frac{2}{3x^2}\right)dx = \frac{1}{12}x^6 + \frac{2}{3x} + C$

e) $\int f(x)\,dx = \int \left(\frac{1}{2}x^3 - 2x^2 + 4x - 5\right)dx = \frac{1}{8}x^4 - \frac{2}{3}x^3 + 2x^2 - 5x + C$

f) $\int f(x)\,dx = \int \left(x^{10} - \frac{1}{2}x^3 - x\right)dx = \frac{1}{11}x^{11} - \frac{1}{8}x^4 - \frac{1}{2}x^2 + C$

16.3 a) $F_1(x) = \frac{10}{3}x^3 + \frac{1}{2}x^2 + 3$ $F_2(x) = \frac{10}{3}x^3 + \frac{1}{2}x^2 - 1$

b) $F_1(x) = \frac{1}{4}x^4 + \frac{3}{2}x^2 + x - 7$ $F_2(x) = \frac{1}{4}x^4 + \frac{3}{2}x^2 + x - 100$

Hints:
- First, determine the indefinite integral of f.
- Then, determine the value of the integration constant such that the stated conditions are fulfilled.

16.4 a) $f(x) = x^3 - 25x^2 + 250x + 500$

b) $f(x) = x^3 - 25x^2 + 250x + 1500$

16.5 $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x - \frac{17}{6}$

16.6 $C(x) = (x^2 + 100x + 200)$ CHF

Hints:
- First integrate the marginal cost function $C'(x) \Rightarrow C(x) = (x^2 + 100x + C)$ CHF $(C \in \mathbb{R})$
- Determine the integration constant C using the fact that $C(0) = 200$ CHF $\Rightarrow C = 200$

16.7 $C(x) = (2x^2 + 2x + 80)$ CHF

16.8 $C(30) = 3750$ CHF

Hint:
- First, determine the cost function $C(x) \Rightarrow C(x) = (2x^2 + 40x + 750)$ CHF.

16.9 (see next page)

16.9 a) $P(x) = (-4x^2 + 24x - 20)$ CHF

Hints:
- First, determine the cost and revenue functions C(x) and R(x).
$\Rightarrow$ $C(x) = \left(\frac{3}{2}x^2 + 20x + 20\right)$ CHF
$R(x) = \left(-\frac{5}{2}x^2 + 44x\right)$ CHF
- Then, determine the profit function P(x).

b) 3 units

Hints:
- The profit function P(x) is a quadratic function.
- Think of the graph of the profit function when determining the global maximum.

16.10 a) $P(x) = \left(-\frac{1}{15}x^3 + 11x^2 - 200x - 10'000\right)$ CHF

Hints:
- First, determine the revenue function R(x) $\Rightarrow$ $R(x) = 400x$ CHF
- Then, determine the average cost function $\overline{C}(x)$ $\Rightarrow$ $\overline{C}(x) = \left(\frac{1}{15}x^2 - 11x + \frac{10'000}{x} + C\right)$ CHF
- Then, determine the total cost function C(x) $\Rightarrow$ $C(x) = \left(\frac{1}{15}x^3 - 11x^2 + 600x + 10'000\right)$ CHF
- Finally, determine the profit function P(x) $\Rightarrow$ $P(x) = R(x) - C(x) = \ldots$

b) 100 units

Hints:
- Determine the local maxima of the profit function P(x).
- Check if one of the local maxima is the global maximum.

c) $P_{max} = P(100) = 13'333$ CHF (rounded)

16.11 a) 2nd statement

b) 3rd statement

c) 2nd statement