

Exercises 14 **Differentiation rules** **Coefficient, sum, product, exponential function, higher-order derivatives**

Objectives

- be able to apply the coefficient, sum, and product rules to determine the derivative of a function.
- be able to determine a higher-order derivative of a function.

Problems

14.1 Determine the derivative by applying the **coefficient rule**:

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|----|----------------------------------|----|---------------------------|----|----------------------|
| a) | $f(x) = 3x^5$ | b) | $f(x) = -4x^3$ | c) | $f(x) = -x^{10}$ |
| d) | $f(x) = a \cdot x^3$ | e) | $f(x) = n \cdot x^{n-1}$ | f) | $f(x) = 9 \cdot 3^x$ |
| g) | $s(t) = \frac{1}{2} g \cdot t^2$ | h) | $S(T) = \alpha \cdot T^4$ | i) | $C(x) = (-3x)^3$ |

14.2 Determine the derivative by applying the **sum rule**:

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|----|--|----|---------------------------------------|----|--|
| a) | $f(x) = x^5 + x^6$ | b) | $f(x) = x^{10} - x^9$ | c) | $f(x) = 1 + x + 3x^3$ |
| d) | $f(x) = \frac{1}{4}x^4 + 3x^2 - 2$ | e) | $f(x) = 3x^2(x - 2)$ | f) | $f(x) = -3x^8 + x^5 - 3x + 99$ |
| g) | $f(x) = ax^2 + bx + c$ | h) | $f(x) = 3(a^2 - 2ax + x^2)$ | i) | $f(x) = \frac{x^3}{3} - \frac{3}{x^3}$ |
| j) | $s(t) = s_0 + v_0t + \frac{1}{2}g \cdot t^2$ | k) | $V(r) = -\frac{a}{r} + \frac{b}{r^2}$ | l) | $C(n) = C_0(1 + nr)$ |

Hint:

- In some problems, the coefficient rule is needed, too.

14.3 Determine the derivative by applying the **product rule**:

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|----|----------------------------------|----|---|
| a) | $f(x) = x \cdot e^x$ | b) | $f(x) = x^3 \cdot 3^x$ |
| c) | $f(x) = -2x^5(x - 1)$ | d) | $f(x) = (2x - 1) \cdot e^x$ |
| e) | $f(x) = (2x - 1)(-3x^2 - x + 1)$ | f) | $V(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3} \right)$ |

Hint:

- In some problems, the coefficient and/or the sum rule(s) is/are needed, too.

14.4 Determine the derivative of the exponential functions below:

- | | | | |
|----|-------------------|----|-----------------------|
| a) | $f(x) = e^{4x}$ | b) | $f(x) = e^{-x}$ |
| c) | $f(x) = e^{-x^2}$ | d) | $f(x) = e^{x^2-2x+5}$ |

14.5 Determine the derivative of the functions below. Apply the appropriate differentiation rule(s). Simplify and factorise the derivative as far as possible:

- | | | | |
|----|--|----|---------------------------|
| a) | $f(x) = (x - 2) e^{2x}$ | b) | $f(x) = (2 - x^2) e^{-x}$ |
| c) | $f(x) = (3x^3 - 2x^2 + x - 1) e^{-2x}$ | d) | $P(v) = av^2 e^{-bv^2}$ |

14.6 (see next page)

14.6 Determine the derivative (rate of change) of the functions below at the indicated position:

- a) f in 14.1 b) $x = 2$ b) s in 14.1 g) $t = 4$
c) f in 14.2 g) $x = -1$ d) P in 14.5 d) $v = 1$

14.7 Determine the second and third derivatives of the functions below. Simplify and factorise the higher-order derivatives as far as possible:

- a) f in 14.1 a) b) f in 14.2 g)
c) f in 14.3 a) d) f in 14.4 c)

14.8 Determine the indicated higher-order derivatives:

- a) $f''(-1)$ with function f in 14.1 a)
Hint:
- You have already determined $f'(x)$ in 14.7 a).
b) $f'''(2)$ with function f in 14.4 c)
Hint:
- You have already determined $f''(x)$ in 14.7 d).

14.9 Decide which statements are true or false. Put a mark into the corresponding box. In each problem a) to c), exactly one statement is true.

- a) The third derivative of a function is a ...
 ... constant function if the second derivative is a quadratic function.
 ... quadratic function if the second derivative is a linear function.
 ... linear function if the first derivative is a quadratic function.
 ... constant function if the first derivative is a quadratic function.
- b) The derivative of a ...
 ... product is the product of the derivatives of the single factors.
 ... product is the sum of the derivatives of the single factors.
 ... sum is the sum of the derivatives of the single addends.
 ... constant is the constant itself.
- c) If $f(x) = c \cdot g(x) \cdot h(x)$ then $f'(x) = \dots$
 ... 0
 ... $c \cdot g'(x) \cdot h'(x)$
 ... $c \cdot g(x) \cdot h'(x) + c \cdot g'(x) \cdot h(x)$
 ... $c \cdot g'(x) \cdot h'(x) + c \cdot g(x) \cdot h(x)$

Answers

- 14.1 a) $f'(x) = 3 \cdot 5x^4 = 15x^4$
 b) $f'(x) = (-4) 3x^2 = -12x^2$
 c) $f'(x) = (-1) 10x^9 = -10x^9$
 d) $f'(x) = a \cdot 3x^2 = 3ax^2$

Hint:

- a is a constant.

- e) $f'(x) = n(n-1)x^{n-2}$
 f) $f'(x) = 9 \cdot 3^x \cdot \ln(3)$
 g) $s'(t) = \frac{g}{2} 2t = gt$

Hints:

- The name of the function is s, and the variable is t.

- g is a constant.

- h) $S'(T) = \alpha \cdot 4T^3 = 4\alpha T^3$
 i) $C'(x) = -81x^2$

- | | | | |
|------|--------------------------|---|----------------------------------|
| 14.2 | a) $f'(x) = 5x^4 + 6x^5$ | b) $f'(x) = 10x^9 - 9x^8$ | c) $f'(x) = 1 + 9x^2$ |
| | d) $f'(x) = x^3 + 6x$ | e) $f'(x) = 9x^2 - 12x$ | f) $f'(x) = -24x^7 + 5x^4 - 3$ |
| | g) $f'(x) = 2ax + b$ | h) $f'(x) = -6a + 6x$ | i) $f'(x) = x^2 + \frac{9}{x^4}$ |
| | j) $s'(t) = v_0 + gt$ | k) $V'(r) = \frac{a}{r^2} - \frac{2b}{r^3}$ | l) $C'(n) = C_0 r$ |

- 14.3 a) $f'(x) = e^x + x \cdot e^x$
 b) $f'(x) = 3x^2 \cdot 3^x + x^3 \cdot 3^x \cdot \ln(3)$
 c) $f'(x) = -2(5x^4(x-1) + x^5)$
 d) $f'(x) = 2 \cdot e^x + (2x-1) \cdot e^x$
 e) $f'(x) = 2(-3x^2 - x + 1) + (2x-1)(-6x-1)$
 f) $V'(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3} \right) + e^r \left(2a \cdot r + \frac{3b}{r^4} \right)$

Hints:

- V is the name of the function, and r is the variable.

- a and b are constants.

- | | | |
|------|---------------------------------|------------------------------------|
| 14.4 | a) $f'(x) = 4 e^{4x}$ | b) $f'(x) = (-1) e^{-x} = -e^{-x}$ |
| | c) $f'(x) = -2x \cdot e^{-x^2}$ | d) $f'(x) = (2x-2) e^{x^2-2x+5}$ |

- 14.5 a) $f'(x) = e^{2x} + (x-2) 2 e^{2x} = (2x-3) e^{2x}$
 b) $f'(x) = -2x e^{-x} + (2-x^2)(-1) e^{-x} = (x^2-2x-2) e^{-x}$
 c) $f'(x) = (9x^2-4x+1) e^{-2x} + (3x^3-2x^2+x-1)(-2) e^{-2x} = (-6x^3+13x^2-6x+3) e^{-2x}$
 d) $P'(v) = a \left(2v e^{-bv^2} + v^2(-2bv) e^{-bv^2} \right) = 2av(1-bv^2) e^{-bv^2}$

14.6 (see next page)

- 14.6 a) 14.1 b)
 $f'(2) = -48$
- b) 14.1 g)
 $s'(4) = 4g$
- c) 14.2 g)
 $f'(-1) = -2a + b$
- d) 14.5 d)
 $P'(1) = 2a(1 - b)e^{-b}$
- 14.7 a) 14.1 a)
 $f''(x) = 15 \cdot 4x^3 = 60x^3$
 $f'''(x) = 60 \cdot 3x^2 = 180x^2$
- b) 14.2 g)
 $f''(x) = 2a \cdot 1 = 2a$
 $f'''(x) = 0$
- c) 14.3 a)
 $f''(x) = e^x + (e^x + x \cdot e^x) = (x + 2) e^x$
 $f'''(x) = e^x + (x + 2) e^x = (x + 3) e^x$
- d) 14.4 c)
 $f''(x) = -2(e^{-x^2} + x(-2x)e^{-x^2}) = 2(2x^2 - 1)e^{-x^2}$
 $f'''(x) = 2(4xe^{-x^2} + (2x^2 - 1)(-2x)e^{-x^2}) = 4x(-2x^2 + 3)e^{-x^2}$
- 14.8 a) $f''(-1) = 60(-1)^3 = -60$
- b) $f'''(2) = 4 \cdot 2(-2 \cdot 2^2 + 3)e^{-2^2} = -\frac{40}{e^4}$
- 14.9 a) 4th statement
- b) 3rd statement
- c) 3rd statement