Exercises 17 Definite integral
Definite integral, area under a curve, consumer's/producer's surplus

Objectives

- be able to apply the fundamental theorem of calculus.
- be able to determine a definite integral of a constant, basic power, and basic exponential function.
- be able to determine the area between the graph of a basic power function and the abscissa.
- be able to determine a consumer's and a producer's surplus if the demand and supply functions are basic power functions.

Problems

17.1 Calculate the definite integrals below:

a) $\int_3^4 (2x - 5)\, dx$ b) $\int_0^1 (x^3 + 2x)\, dx$ c) $\int_{-5}^{-3} \left(\frac{x^2}{2} - 4\right) dx$

d) $\int_2^4 \left(x^3 - \frac{x^2}{2} + 3x - 4\right) dx$ e) $\int_{-2}^{2} \left(2x^2 - \frac{x^4}{8}\right) dx$ f) $\int_{-1}^{1} e^x\, dx$

g) $\int_0^1 e^{2x}\, dx$ h) $\int_{-1}^{1} e^{-3x}\, dx$

17.2 Determine the area between the graph of the function f and the x-axis on the interval where the graph of f is above the x-axis, i.e. where $f(x) \geq 0$.

a) $f(x) = -x^2 + 1$ b) $f(x) = x^3 - x^2 - 2x$

17.3 The demand function (price in CHF) for a product is $p = f(x) = 100 - 4x^2$.
If the equilibrium quantity is 4 units, what is the consumer's surplus?

17.4 The demand function (price in CHF) for a product is $p = f(x) = 34 - x^2$.
If the equilibrium price is 9 CHF, what is the consumer's surplus?

17.5 The demand function (price in CHF) for a certain product is
$p = f(x) = 81 - x^2$
and the supply function (price in CHF) is
$p = g(x) = x^2 + 4x + 11$.

Determine …

a) … the equilibrium point, i.e. the equilibrium quantitiy and the equilibrium price.

b) … the consumer's surplus at market equilibrium.

c) … the producer’s surplus at market equilibrium.

17.6 Suppose that the supply function (price in CHF) for a good is $p = g(x) = 4x^2 + 2x + 2$.
If the equilibrium price is 422 CHF, what is the producer's surplus?

17.7 The demand function (price in CHF) for a certain product is
$p = f(x) = 144 - 2x^2$
and the supply function (price in CHF) is
$p = g(x) = x^2 + 33x + 48$

Determine the producer's surplus at the equilibrium point.

17.8 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) The definite integral of a function is a ...

- ☐ ... real number.
- ☐ ... function.
- ☐ ... set of functions.
- ☐ ... graph.

b) $\int_a^b f(x)\,dx$ ...

- ☐ ... = f(b) – f(a)
- ☐ ... = F(a) – F(b) where F is an antiderivative of f.
- ☐ ... is equal to the area between the graph of f and the x-axis in the interval [a,b] if $f(x) \geq 0$ for all $x \in [a,b]$
- ☐ ... cannot be calculated unless all antiderivatives of f are known.

c) The consumer's surplus is an area between ...

- ☐ ... the graphs of the demand and the supply functions.
- ☐ ... the x axis and the graph of the demand function.
- ☐ ... the graph of the demand function and the horizontal line "price = equilibrium price".
- ☐ ... the horizontal line "price = equilibrium price" and the graph of the supply function.

Answers

17.1 a) $\int_3^4 (2x - 5)\,dx = [x^2 - 5x]_3^4 = (4^2 - 5\cdot 4) - (3^2 - 5\cdot 3) = 2$

b) $\int_0^1 (x^3 + 2x)\,dx = \left[\frac{x^4}{4} + x^2\right]_0^1 = \left(\frac{1^4}{4} + 1^2\right) - \left(\frac{0^4}{4} + 0^2\right) = \frac{5}{4}$

c) $\int_{-5}^{-3} \left(\frac{x^2}{2} - 4\right) dx = \left[\frac{x^3}{6} - 4x\right]_{-5}^{-3} = \left(\frac{(-3)^3}{6} - 4\cdot(-3)\right) - \left(\frac{(-5)^3}{6} - 4\cdot(-5)\right) = \frac{25}{3}$

d) $\int_2^4 \left(x^3 - \frac{x^2}{2} + 3x - 4\right) dx = \left[\frac{x^4}{4} - \frac{x^3}{6} + \frac{3x^2}{2} - 4x\right]_2^4 = \left(\frac{4^4}{4} - \frac{4^3}{6} + \frac{3\cdot 4^2}{2} - 4\cdot 4\right) - \left(\frac{2^4}{4} - \frac{2^3}{6} + \frac{3\cdot 2^2}{2} - 4\cdot 2\right) = \frac{182}{3}$

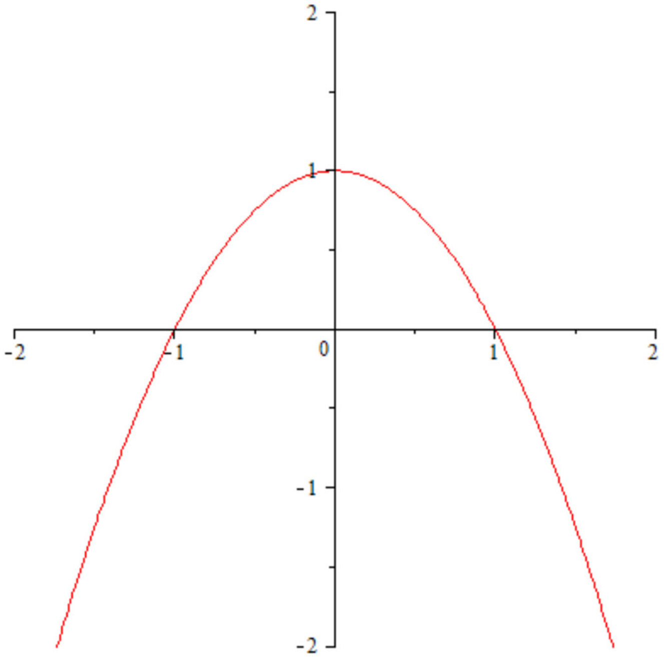
e) $\int_{-2}^{2} \left(2x^2 - \frac{x^4}{8}\right) dx = \left[\frac{2x^3}{3} - \frac{x^5}{40}\right]_{-2}^{2} = \left(\frac{2\cdot 2^3}{3} - \frac{2^5}{40}\right) - \left(\frac{2\cdot(-2)^3}{3} - \frac{(-2)^5}{40}\right) = \frac{136}{15}$

f) $\int_{-1}^{1} e^x\,dx = [e^x]_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}$

g) $\int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}(e^2 - 1)$

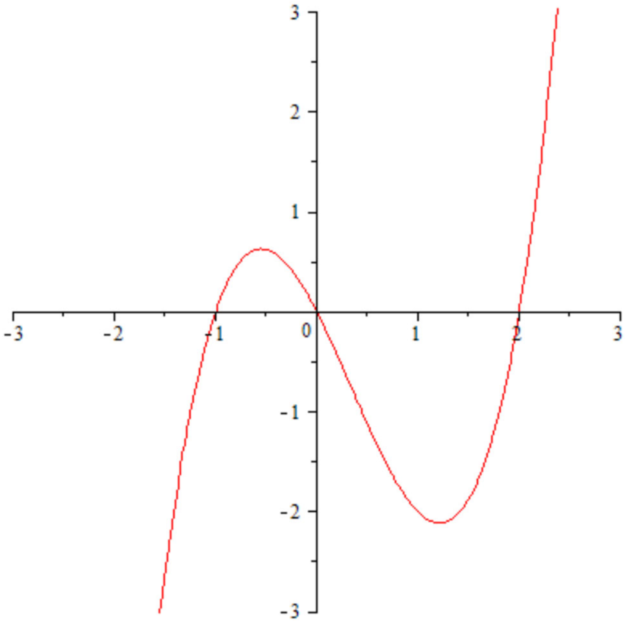
h) $\int_{-1}^{1} e^{-3x}\,dx = \left[-\frac{1}{3}e^{-3x}\right]_{-1}^{1} = -\frac{1}{3}(e^{-3} - e^3) = \frac{1}{3}\left(e^3 - \frac{1}{e^3}\right)$

17.2 a) $A = \int_{-1}^{1} (-x^2 + 1)\,dx = \left[-\frac{x^3}{3} + x\right]_{-1}^{1} = \frac{4}{3}$



b) (see next page)

b) $A = \int_{-1}^{0}(x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = \frac{5}{12}$



Hints:
- First, determine the positions x where the graph of f intersects the x-axis, i.e where $f(x) = 0$
- Then, determine the interval on which the graph of f is above the x-axis, i.e. where $f(x) \geq 0$

17.3 Consumer's surplus CS = 170.67 CHF (rounded)

17.4 Consumer's surplus CS = 83.33 CHF (rounded)

17.5 a) Equilibrium quantity $x = 5$
Equilibrium price $p = 56$ CHF

b) Consumer's surplus CS = 83.33 CHF (rounded)

c) Producer's surplus PS = 133.33 CHF (rounded)

17.6 Producer's surplus PS = 2766.67 CHF (rounded)

17.7 Producer's surplus PS = 103.34 CHF (rounded)

17.8 a) 1st statement

b) 3rd statement

c) 3rd statement