Exercises 12 Exponential function and equations
Ordinary annuity, annuity due

Objectives

- be able to calculate the present and the future value of an annuity if constant payments are made at the beginning or at the end of each compounding period.
- be able to treat specific annuity tasks.

Problems

Ordinary annuity

12.1 Find the future value of an annuity of 1300 CHF paid at the end of each year for 5 years, if interest is earned at a rate of 6%, compounded annually.

12.2 The formula

$$A_n = p\,\frac{q^n - 1}{q - 1}$$

is used for calculating the future value A_n of an ordinary annuity.

Solve the formula for p and n.

12.3 A family wants to have a 200'000 CHF college fund for their children at the end of 20 years. What contribution must be made at the end of each quarter if their investment pays 7.6%, compounded quarterly?

12.4 If 2500 CHF is deposited at the end of each quarter in an account that earns 5% compounded quarterly, after how many quarters will the account contain 80'000 CHF?

12.5 Assume that money on a savings account pays 1.5%, compounded annually. In order to have 20'000 CHF at the end of 10 years, ...

a) ... what payment must be made at the end of each year?

b) ... what amount has to be paid in at the beginning of the ten years if no more payments are made for the rest of the time?

c) Compare the answers in a) and b), and explain why the payment made in b) is a smaller amount than the sum of the 10 payments made in a).

12.6 Two twins are 23 years old and have different investment strategies.

Suppose that twin 1 invests 2000 CHF at the end of each year for 10 years only (until age 33) in an account that earns 8%, compounded annually. Suppose that twin 2 waits until turning 40 to begin investing.

How much must twin 2 put aside at the end of each year for the next 25 years in an account that earns 8%, compounded annually, in order to have the same amount as twin 1 when he turns 65?

Hints:
- Draw a diagram which shows the investment strategies of the two twins with respect to time.
- The money twin 1 has paid in by the time he turns 33 pays interest until he turns 65.

12.7 Find the initial value of an annuity if 6000 CHF can be withdrawn at the end of each 6-month period for 8 years and if fhe interest rate is 8%, compounded semiannually.

12.8 The formula

$$A_0 = p\,\frac{q^n - 1}{q^n(q - 1)}$$

is used for calculating the initial value A_0 of an ordinary annuity.

Solve the formula for p and n.

12.9 With an initial value of 135'000 CHF, what is the size of the withdrawals that can be made at the end of each quarter for the next 10 years if money is worth 6.4%, compounded quarterly?

12.10 A personal account earmarked as a retirement supplement contains 242'000 CHF. Suppose 200'000 CHF is used to establish an annuity that earns 6%, compounded quarterly, and pays 4500 CHF at the end of each quarter. How long will it be until the account balance is 0 CHF?

12.11 Mr. X owns a caravan. He could sell it now for 20'000 CHF. Alternatively, he could rent it out for 10 years at 2100 CHF per year, the rent being paid at the end of each year. After 10 years the caravan would be completely depreciated. Mr. X could invest the revenues (for either selling or renting out the caravan) at 7%, compounded annually. Which alternative is more beneficial for Mr. X?

12.12 In a building loan contract 3600 CHF are paid in at the end of each year. The money earns interest at an annual rate of 3%. After 10 years twice the saved money is paid out. The debts, worth 5%, compounded annually, have to be paid off within 10 years by instalments due at the end of each year. What is the size of the annual payments (in order to pay off the debts)?

Annuity due

12.13 The two formulae

$$A_n = pq\,\frac{q^n - 1}{q - 1} \qquad \text{and} \qquad A_0 = p\,\frac{q^n - 1}{q^{n-1}(q - 1)}$$

are used for calculating the future value A_n or the initial value A_0 of an annuity due.

Solve both formulae for p and n.

12.14 Find the future value of an annuity due of 100 CHF each quarter for 2.5 years at 12%, compounded quarterly.

12.15 How much must be deposited at the beginning of each year in an account that pays 8%, compounded annually, so that the account will contain 24'000 CHF at the end of 5 years?

12.16 If an account that earns 5%, compounded quarterly, contains 80'000 CHF at the beginning and 2500 CHF is withdrawn at the beginning of each quarter, after how many quarters will the account contain 0 CHF?

12.17 What amount must be set aside now to generate payments of 50'000 CHF at the beginning of each year for the next 12 years if money is worth 5.92%, compounded annually?

12.18 A year-end bonus of 25'000 CHF will generate how much money at the beginning of each month for the next year, if it can be invested at 6.48%, compounded monthly?

Miscellaneous problems

12.19 A couple has determined that they need 300'000 CHF to establish an annuity when they retire in 25 years. How much money should they deposit at the end of each month in an investment plan that pays 10%, compounded monthly, so they will have the 300'000 CHF in 25 years?

12.20 Mr. Gordon plans to invest 300 CHF at the end of each month in an account that pays 9%, compounded monthly. After how many months will the account be worth 50'000 CHF?

12.21 Grandparents plan to open an account on their grandchild's birthday and contribute each month until she goes to college. How much must they contribute at the beginning of each month in an investement that pays 12%, compounded monthly, if they want the balance to be 180'000 CHF at the end of 18 years?

12.22 An insurance settlement of 750'000 CHF must replace somebody’s income for the next 40 years. What income will this settlement provide at the end of each month if it is invested in an annuity that earns 8.4%, compounded monthly?

12.23 Sara's parents want to establish a college trust for her. They want to make 16 quarterly withdrawals of 2000 CHF, with the first withdrawal 3 months from now. If money is worth 7.2%, compounded quarterly, how much must be deposited now to provide for this trust?

12.24 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) In an ordinary annuity scheme ...

- ☐ ... money is always paid in or withdrawn once per year.
- ☐ ... money is paid in or withdrawn at the beginning of each period.
- ☐ ... the value of the annuity grows or decays exponentially.
- ☐ ... no payments are made during a compounding period.

b) In an ordinary annuity scheme interest is compounded monthly. If 100 CHF are paid in each month it can be concluded that the value of the annuity after one year is ...

- ☐ ... 1200 CHF.
- ☐ ... 1320 CHF if the annual interest rate is 10%.
- ☐ ... less than 1320 CHF if the annual interest rate is 10%.
- ☐ ... less than 1200 CHF.

c) Assume an initial capital of 1000 CHF. In an annuity due scheme (interest rate = 1%, compounded annually) a constant amount of money should be withdrawn 10 times at the beginning of each year. Therefore, the annual withdrawals ...

- ☐ ... must not be greater than 100 CHF.
- ☐ ... must be exactly 100 CHF.
- ☐ ... could be twice as high if the interest rate equalled 2%.
- ☐ ... could be greater in an ordinary annuity scheme.

Answers

12.1 $A_n = p\,\frac{q^n - 1}{q - 1}$ where p = 1300 CHF, q = 1 + 6% = 1.06, n = 5

$\Rightarrow$ A_5 = 7328.22 CHF (rounded)

12.2 see formulary

12.3 $p = \frac{A_n(q - 1)}{q^n - 1}$ where A_n = 200'000 CHF, $q = 1 + \frac{7.6\%}{4}$, $n = 20 \cdot 4 = 80$

$\Rightarrow$ p = 1083.40 CHF (rounded)

12.4 $n = \frac{\lg\left(\frac{A_n(q - 1)}{p} + 1\right)}{\lg(q)}$ where A_n = 80'000 CHF, p = 2500 CHF, $q = 1 + \frac{5\%}{4}$

$\Rightarrow$ n = 27.08... $\rightarrow$ 28 quarters = 7 years

12.5 a) Ordinary annuity

$p = \frac{A_n(q - 1)}{q^n - 1}$ where A_n = 20'000 CHF, q = 1 + 1.5%, n = 10

$\Rightarrow$ p = 1868.70 CHF (rounded up)

b) Compound interest

$C_0 = \frac{C_n}{q^n}$ where C_n = 20'000 CHF, q = 1 + 1.5%, n = 10

$\Rightarrow$ C_0 = 17'233.35 CHF (rounded up)

c) …

12.6 Twin 1: Ordinary annuity (from age 23 to age 33)

$A_n = p\,\frac{q^n - 1}{q - 1}$ where p = 2000 CHF, q = 1 + 8%, n = 10

$\Rightarrow$ A_{10} = capital at the age of 33 = 28'973.12 CHF (rounded)

Compound interest (from age 33 to age 65)

$C_n = C_0\,q^n$ where $C_0 = A_{10}$, q = 1 + 8%, n = 32

$\Rightarrow$ C_{32} = capital at the age of 65 = 340'059.97 CHF (rounded)

(C_{32} = capital of twin 2 at the age of 65)

Twin 2: Ordinary annuity (from age 40 to age 65)

$p = \frac{A_n(q - 1)}{q^n - 1}$ where $A_n = C_{32}$ (twin 1) = 340'059.97 CHF, q = 1 + 8%, n = 25

$\Rightarrow$ p = 4651.61 CHF (rounded)

12.7 $A_0 = p\,\frac{q^n - 1}{q^n(q - 1)}$ where p = 6000 CHF, $q = 1 + \frac{8\%}{2}$, $n = 8 \cdot 2 = 16$

$\Rightarrow$ A_0 = 69'913.77 CHF (rounded)

12.8 see formulary

12.9 $p = \frac{A_0\, q^n (q-1)}{q^n - 1}$ where A_0 = 135'000 CHF, $q = 1 + \frac{6.4\%}{4}$, $n = 10 \cdot 4 = 40$

$\Rightarrow$ p = 4595.46 CHF (rounded)

12.10 $n = \frac{\lg\left(\frac{p}{p - A_0(q-1)}\right)}{\lg(q)}$ where A_0 = 200'000 CHF, p = 4500 CHF, $q = 1 + \frac{6\%}{4}$

$\Rightarrow$ n = 73.78... $\rightarrow$ 73 quarters (less than 4500 CHF at the end of the 74th quarter)

12.11 Alternative 1 (selling the caravan): Compound interest

$C_n = C_0\, q^n$ where C_0 = 20'000 CHF, q = 1 + 7%, n = 10

$\Rightarrow$ C_{10} = 39'343 CHF (rounded)

Alternative 2 (renting out the caravan): Ordinary annuity

$A_n = p\, \frac{q^n - 1}{q - 1}$ where p = 2100 CHF, q = 1 + 7%, n = 10

$\Rightarrow$ A_{10} = 29'015 CHF (rounded)

$\Rightarrow$ $C_{10} > A_{10}$, i.e. alternative 1 is more beneficial

12.12 2 annuities: first 10 years (paying in money), second 10 years (paying off the debts)

- first 10 years (saving money, i.e. paying in money)

$A_n = p\, \frac{q^n - 1}{q - 1}$ where p = 3600 CHF, q = 1 + 3%, n = 10

$\Rightarrow$ A_{10} = 41'270 CHF (rounded)

- second 10 years (paying off debts)

$p = \frac{A_0\, q^n (q-1)}{q^n - 1}$ where A_0 = 41'270 CHF, q = 1 + 5%, n = 10

$\Rightarrow$ p = 5345 CHF (rounded)

12.13 see formulary

12.14 $A_n = pq\, \frac{q^n - 1}{q - 1}$ where p = 100 CHF, $q = 1 + \frac{12\%}{4}$, $n = 2.5 \cdot 4 = 10$

$\Rightarrow$ A_{10} = 1180.78 CHF (rounded)

12.15 $p = \frac{A_n(q-1)}{q(q^n - 1)}$ where A_n = 24'000 CHF, q = 1 + 8%, n = 5

$\Rightarrow$ p = 3787.92 CHF (rounded)

12.16 $n = \frac{\lg\left(\frac{pq}{pq - A_0(q-1)}\right)}{\lg(q)}$ where A_0 = 80'000 CHF, p = 2500 CHF, $q = 1 + \frac{5\%}{4}$

$\Rightarrow$ n = 40.46... $\rightarrow$ 40 quarters (less than 2500 CHF at the beginning of the 41st quarter)

12.17 $A_0 = p\, \frac{q^n - 1}{q^{n-1}(q - 1)}$ where p = 50'000 CHF, q = 1 + 5.92%, n = 12

$\Rightarrow$ A_0 = 445'962.23 CHF (rounded)

12.18 $p = \frac{A_0\, q^{n-1}(q-1)}{q^n - 1}$ where $A_0 = 25'000$ CHF, $q = 1 + \frac{6.48\%}{12}$, $n = 1 \cdot 12 = 12$

$\Rightarrow$ $p = 2145.59$ CHF (rounded)

12.19 Ordinary annuity

$p = \frac{A_n(q-1)}{q^n - 1}$ where $A_n = 300'000$ CHF, $q = 1 + \frac{10\%}{12}$, $n = 25 \cdot 12 = 300$

$\Rightarrow$ $p = 226.10$ CHF (rounded)

12.20 Ordinary annuity

$n = \frac{\lg\left(\frac{A_n(q-1)}{p} + 1\right)}{\lg(q)}$ where $A_n = 50'000$ CHF, $p = 300$ CHF, $q = 1 + \frac{9\%}{12}$

$\Rightarrow$ $n = 108.52... \rightarrow$ 109 months = 9 years 1 month

12.21 Annuity due

$p = \frac{A_n(q-1)}{q(q^n - 1)}$ where $A_n = 180'000$ CHF, $q = 1 + \frac{12\%}{12}$, $n = 18 \cdot 12 = 216$

$\Rightarrow$ $p = 235.16$ CHF (rounded)

12.22 Ordinary annuity, income = monthly payment p

$p = \frac{A_0\, q^n(q-1)}{q^n - 1}$ where $A_0 = 750'000$ CHF, $q = 1 + \frac{8.4\%}{12}$, $n = 40 \cdot 12 = 480$

$\Rightarrow$ $p = 5441.23$ CHF (rounded)

12.23 Ordinary annuity

$A_0 = p\, \frac{q^n - 1}{q^n(q-1)}$ where $p = 2000$ CHF, $q = 1 + \frac{7.2\%}{4}$, $n = 16$

$\Rightarrow$ $A_0 = 27'590.62$ CHF (rounded)

12.24 a) 4th statement

b) 3rd statement

c) 4th statement