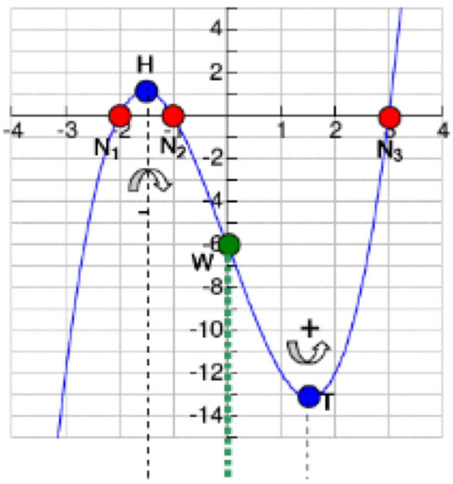
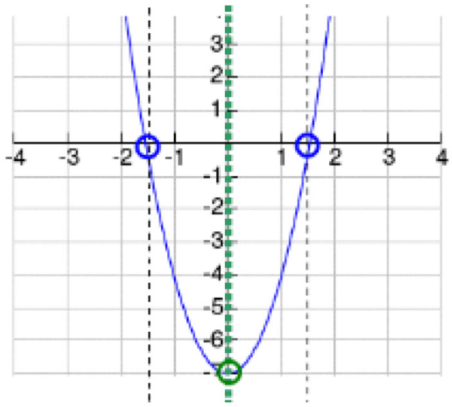
Increasing/decreasing, concavity

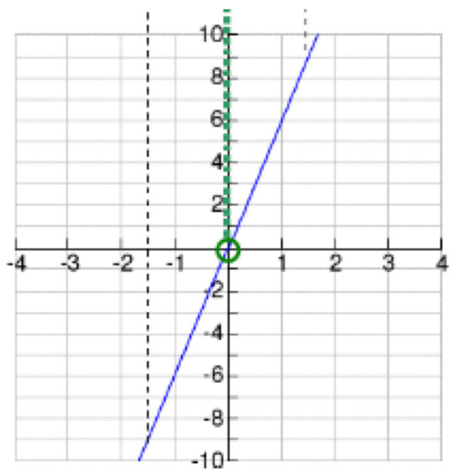
Ex.: $f(x) = x^3 - 7x - 6$



$f'(x) = 3x^2 - 7$



$f''(x) = 6x$



Increasing/decreasing

If the **first derivative** of the function f is **positive** at $x = x_0$, i.e. $f'(x_0) > 0$, f is **increasing** at $x = x_0$.

If the **first derivative** of the function f is **negative** at $x = x_0$, i.e. $f'(x_0) < 0$, f is **decreasing** at $x = x_0$.

Concavity

If the **second derivative** of the function f is **positive** at $x = x_0$, i.e. $f''(x_0) > 0$, the graph of f is **concave up** ("left-hand bend") at $x = x_0$.

If the **second derivative** of the function f is **negative** at $x = x_0$, i.e. $f''(x_0) < 0$, the graph of f is **concave down** ("right-hand bend") at $x = x_0$.

Relative maxima/minima

The function f has a **relative maximum** at $x = x_0$ if the tangent to the graph of f at $x = x_0$ is horizontal and if the graph of f is concave down at $x = x_0$.

This applies if $f'(x_0) = 0$ (necessary) and $f''(x_0) < 0$ (sufficient).

The function f has a **relative minimum** at $x = x_0$ if the tangent to the graph of f at $x = x_0$ is horizontal and if the graph of f is concave up at $x = x_0$.

This applies if $f'(x_0) = 0$ (necessary) and $f''(x_0) > 0$ (sufficient).

Absolute maximum/minimum

The **absolute maximum/minimum** of a continuous function f is either a relative maximum/minimum or the value of f at one of the endpoints of the domain.

Points of inflection

The function f has a **point of inflection** at $x = x_0$ if the graph of f changes its concavity from concave up to concave down (or vice versa) at $x = x_0$.

This applies if $f''(x_0) = 0$ (necessary) and $f'''(x_0) \neq 0$ (sufficient).

Ex.: $f(x) = x^3 - 7x - 6$ (see page 1) $\Rightarrow f'(x) = 3x^2 - 7$

$\Rightarrow f''(x) = 6x$

$\Rightarrow f'''(x) = 6$

Relative maxima/minima

$f'(x) = 0$ at $x_1 = \sqrt{\frac{7}{3}} = 1.52\ldots$ and $x_2 = -\sqrt{\frac{7}{3}} = -1.52\ldots$

$f''(x_1) = 6 \cdot \sqrt{\frac{7}{3}} = 9.16\ldots > 0$ $\Rightarrow$ relative minimum at $x_1 = \sqrt{\frac{7}{3}}$

$f''(x_2) = -6 \cdot \sqrt{\frac{7}{3}} = -9.16\ldots < 0$ $\Rightarrow$ relative maximum at $x_2 = -\sqrt{\frac{7}{3}}$

Absolute maximum/minimum

Ex.: D = [0,4] ⇒ absolute maximum at x = 4 (endpoint of domain)
⇒ absolute minimum at $x = x_1 = \sqrt{\frac{7}{3}}$ (relative minimum)

Ex.: D = [-4,3] ⇒ absolute maximum at $x = x_2 = -\sqrt{\frac{7}{3}}$ (relative maximum)
⇒ absolute minimum at x = -4 (endpoint of domain)

Points of inflection

$f''(x) = 0$ at $x_3 = 0$

$f'''(x_3) = 6 \neq 0$ ⇒ point of inflection at $x_3 = 0$

Financial mathematics

Marginal cost/revenue/profit function = first derivative of the cost/revenue/profit function

Ex.:
Cost function $C(x) = 120 + 2x^2$
⇒ Marginal cost function $C'(x) = 4x$

Revenue function $R(x) = 168x - x^2$
⇒ Marginal revenue function $R'(x) = 168 - 2x$

Profit function $P(x) = R(x) - C(x) = -120 + 168x - 3x^2$
⇒ Marginal profit function $P'(x) = 168 - 6x$

Average cost/revenue/profit function

Average cost function $\overline{C}(x) := \frac{C(x)}{x}$ where C(x) = cost function

Ex.: Cost function $C(x) = 3x^2 + 4x + 2$
⇒ Average cost function $\overline{C}(x) = 3x + 4 + \frac{2}{x}$

Average revenue function $\overline{R}(x) := \frac{R(x)}{x}$ where R(x) = revenue function

Average profit function $\overline{P}(x) := \frac{P(x)}{x}$ where P(x) = profit function