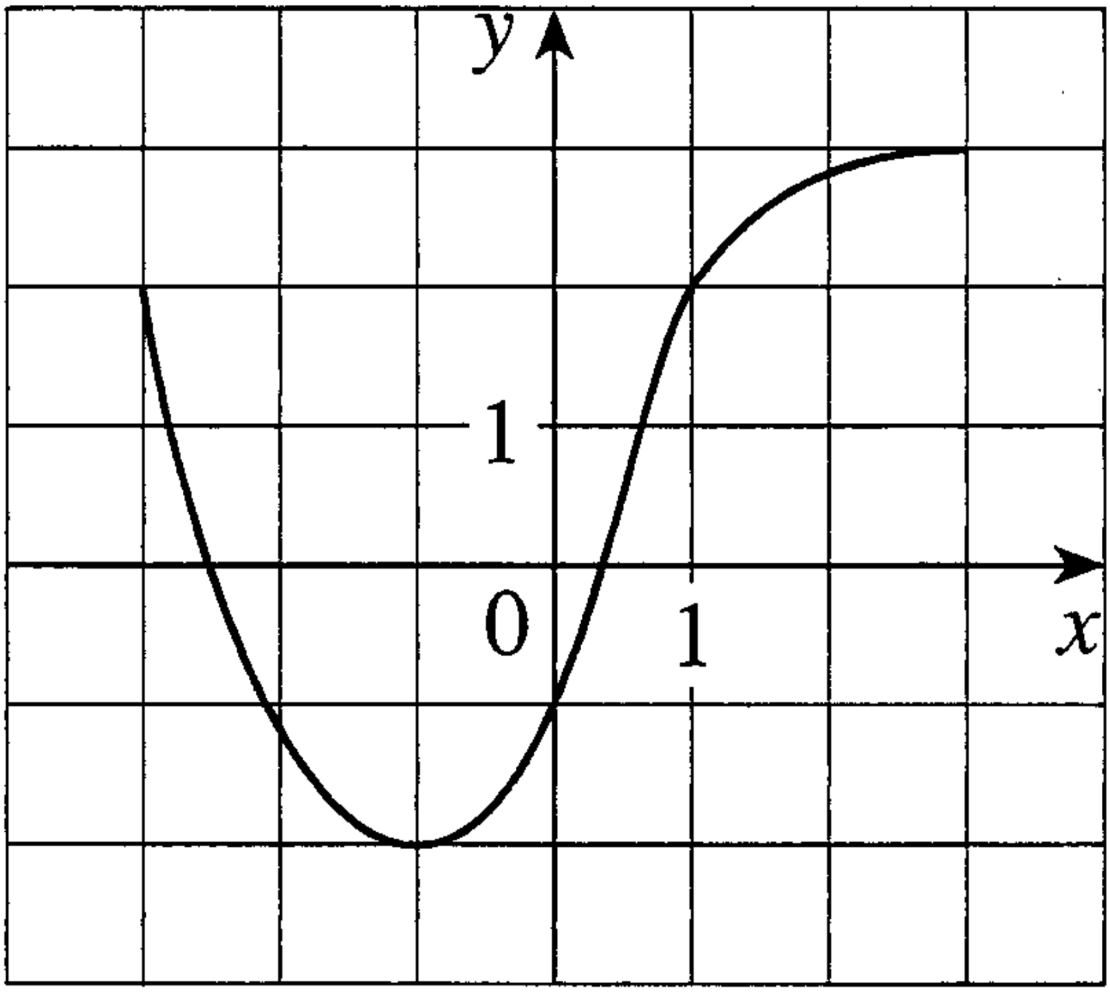
Exercises 13 Derivative
Derivative (rate of change), derivative (derived function) of constant/power/exponential functions

Objectives

- be able to estimate a derivative (rate of change) out of the graph of a function.
- be able to state the derivative (rate of change) of a constant and a linear function.
- be able to determine the derivative (derived function) of a constant and a linear function.
- be able to determine the derivative (derived function) of a basic power and a basic exponential function.
- be able to determine a derivative (rate of change) of a basic power and a basic exponential function.

Problems

13.1 The graph of a function f ist given as follows:



Estimate the derivative (rate of change) $f'(x_0)$ at the given position x_0:

a) $x_0 = -1$ b) $x_0 = 0$

c) $x_0 = 1$ d) $x_0 = -2$

Hints:
- Draw the tangent to the graph of f at the given position x_0.
- Choose two points on the tangent, and estimate their coordinates.
- Determine the slope of the tangent out of the estimated coordinates of the two points.

13.2 For each of the following functions f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = \ldots$

i) ... draw the graph of f.

ii) ... state the derivative (rate of change) $f'(x_0)$ at the given position x_0.

a) $f(x) = 3$ — $x_0 = 2$

b) $f(x) = c \ (c \in \mathbb{R})$ — any $x_0 \in \mathbb{R}$

c) $f(x) = 2x - 3$ — $x_0 = 4$

d) $f(x) = mx + q \ (m \in \mathbb{R}\setminus\{0\}, q \in \mathbb{R})$ — any $x_0 \in \mathbb{R}$

Hint:
- If the graph of a function f is a straight line, the derivative (rate of change) $f'(x_0)$ is the slope of the straight line and does not depend on the position x_0.

13.3 Determine $f'(x)$:

a) $f(x) = 3$ b) $f(x) = 0$ c) $f(x) = -1$

d) $f(x) = x^3$ e) $f(x) = x^4$ f) $f(x) = x^5$

g) $f(x) = x^{17}$ h) $f(x) = x^{200}$ i) $f(x) = x^{100'001}$

j) $f(x) = x^{-1}$ k) $f(x) = x^{-2}$ l) $f(x) = x^{-17}$

m) $f(x) = \frac{1}{x}$ n) $f(x) = \frac{1}{x^3}$ o) $f(x) = \frac{1}{x^{99}}$

p) $f(x) = 3^x$ q) $f(x) = 5^x$ r) $f(x) = \left(\frac{2}{3}\right)^x$

13.4 Determine the derivative (rate of change) $f'(x_0)$ of the function f at the indicated position x_0:

a) $f(x) = x$

i) $x_0 = 0$ ii) $x_0 = 1$ iii) $x_0 = -2$

b) $f(x) = x^5$

i) $x_0 = 0$ ii) $x_0 = 2$ iii) $x_0 = -\frac{2}{3}$

c) $f(x) = x^{-4}$

i) $x_0 = -1$ ii) $x_0 = -\frac{4}{3}$ iii) $x_0 = 0$

d) $f(x) = \left(\frac{2}{3}\right)^x$

i) $x_0 = 0$ ii) $x_0 = 1$ iii) $x_0 = -2$

13.5 * The derivative (rate of change) $f'(x_0)$ of f at the position x_0 can be determined by looking at the secant through the points $A(x_0 \mid f(x_0))$ and $B(x_0+\Delta x \mid f(x_0+\Delta x))$ of the graph of f. The slope of this secant tends towards the slope of the tangent through $A(x_0 \mid f(x_0))$ as Δx tends towards 0.

It has been shown in class how to determine $f'(x_0)$ for the quadratic function $f(x) = x^2$.

Find $f'(x_0)$ for the following functions f:

a) $f(x) = x^3$ b) $f(x) = \frac{1}{x^2}$

13.6 (see next page)

13.6 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) The derivative (rate of change) of a function f at the position x_0 is a ...

- ☐ ... real number.
- ☐ ... function.
- ☐ ... tangent.
- ☐ ... graph.

b) The derivative (derived function) f ' of a function f is a ...

- ☐ ... real number.
- ☐ ... function.
- ☐ ... tangent.
- ☐ ... graph.

c) $f'(x_0)$ is the slope of the ...

- ☐ ... secant through the points $(0|0)$ and $(x_0|f(x_0))$.
- ☐ ... secant through the points $(x_0+\Delta x|f(x_0+\Delta x))$ and $(x_0|f(x_0))$.
- ☐ ... tangent to the graph of f through $(x_0|f(x_0))$.
- ☐ ... tangent to the graph of f ' through $(x_0|f(x_0))$.

Answers

13.1	a)	$f'(-1) \approx 0$	b)	$f'(0) \approx 2$		
	c)	$f'(1) \approx \frac{3}{2}$	d)	$f'(-2) \approx -\frac{5}{3}$		

13.2	a)	i)	...
		ii)	$f'(2) = 0$
	b)	i)	...
		ii)	$f'(x_0) = 0$
	c)	i)	...
		ii)	$f'(4) = 2$
	d)	i)	...
		ii)	$f'(x_0) = m$

13.3	a)	$f'(x) = 0$	b)	$f'(x) = 0$	c)	$f'(x) = 0$
	d)	$f'(x) = 3x^2$	e)	$f'(x) = 4x^3$	f)	$f'(x) = 5x^4$
	g)	$f'(x) = 17x^{16}$	h)	$f'(x) = 200x^{199}$	i)	$f'(x) = 100'001x^{100'000}$
	j)	$f'(x) = -x^{-2}$	k)	$f'(x) = -2x^{-3}$	l)	$f'(x) = -17x^{-18}$
	m)	$f'(x) = -\frac{1}{x^2}$	n)	$f'(x) = -\frac{3}{x^4}$	o)	$f'(x) = -\frac{99}{x^{100}}$
	p)	$f'(x) = 3^x \ln(3)$	q)	$f'(x) = 5^x \ln(5)$	r)	$f'(x) = \left(\frac{2}{3}\right)^x \ln\left(\frac{2}{3}\right)$

13.4	a)	$f'(x) = 1$					
		i)	$f'(0) = 1$	ii)	$f'(1) = 1$	iii)	$f'(-2) = 1$
	b)	$f'(x) = 5x^4$					
		i)	$f'(0) = 0$	ii)	$f'(2) = 80$	iii)	$f'\left(-\frac{2}{3}\right) = \frac{80}{81}$
	c)	$f'(x) = -\frac{4}{x^5}$					
		i)	$f'(-1) = 4$	ii)	$f'\left(-\frac{4}{3}\right) = \frac{243}{256}$	iii)	$f'(0)$ is not defined (division by zero)
	d)	$f'(x) = \left(\frac{2}{3}\right)^x \ln\left(\frac{2}{3}\right)$					
		i)	$f'(0) = \ln\left(\frac{2}{3}\right)$	ii)	$f'(1) = \frac{2}{3}\ln\left(\frac{2}{3}\right)$	iii)	$f'(-2) = \frac{9}{4}\ln\left(\frac{2}{3}\right)$

13.5 *	a)	$f'(x_0) = 3x_0^2$	b)	$f'(x_0) = -\frac{2}{x_0^3}$

13.6	a)	1st statement
	b)	2nd statement
	c)	3rd statement