Exercises 15 Applications of differential calculus
Relative maxima/minima, points of inflection

Objectives

- be able to determine the relative maxima/minima of a function.
- be able to determine the points of inflection of a function.
- be able to determine the absolute maximum/minimum of a cost/revenue/profit function.
- be able to determine the absolute minimum of an average cost function.

Problems

15.1 For each function, determine …

i) … the relative maxima and minima.

ii) … the points of inflection.

a) $f(x) = x^2 - 4$

b) $f(x) = - 8x^3 + 12x^2 + 18x$

c) $s(t) = t^4 - 8t^2 + 16$

d) $f(x) = x\, e^{-x}$

e) * $f(x) = (1 - e^{-2x})^2$

f) * $V(r) = - D\left(\frac{2a}{r} - \frac{a^2}{r^2}\right)$ $(D > 0, a > 0)$

15.2 If the total profit (in CHF) for a commodity is

$P(x) = 2000x + 20x^2 - x^3$

where x is the number of items sold, determine the level of sales, x, that maximises profit, and find the maximum profit.

Hints:
- First, find the relative maximum.
- Then, check if the relative maximum is the absolute maximum.

15.3 If the total cost (in CHF) for a commodity is given by

$C(x) = \frac{1}{4}x^2 + 4x + 100$

where x represents the number of units produced, producing how many units will result in a minimum average cost? Determine the minimum average cost.

15.4 Suppose that the production capacity for a certain commodity cannot exceed 30. If the total profit (in CHF) for this company is

$P(x) = 4x^3 - 210x^2 + 3600x$

where x is the number of units sold, determine the number of items that will maximise profit.

15.5 (see next page)

15.5 Suppose the annual profit for a store (in 1000 CHF) is given by

$$P(x) = - 0.1x^3 + 3x^2$$

where x is the number of years past 2010. If this model is accurate, determine the point of inflection for the profit.

15.6 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) If f has a relative maximum at $x = x_0$ it can be concluded that ...

- ☐ ... $f(x_0) > f(x)$ for any $x \neq x_0$
- ☐ ... $f(x_0) > f(x)$ for any $x > x_0$
- ☐ ... $f(x_0) > f(x)$ for any $x < x_0$
- ☐ ... $f(x_0) > f(x)$ for all x which are in a certain neighbourhood of x_0

b) If $f(x_0) < 0$, $f'(x_0) = 0$, and $f''(x_0) \neq 0$, it can be concluded that f has ...

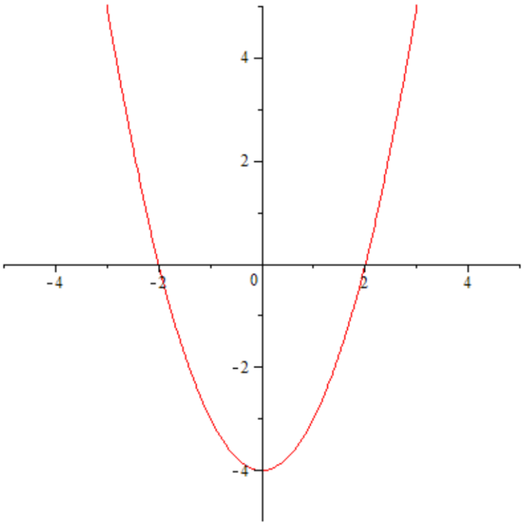
- ☐ ... no relative minimum at $x = x_0$
- ☐ ... no relative maximum at $x = x_0$
- ☐ ... no point of inflection at $x = x_0$
- ☐ ... a point of inflection at $x = x_0$

c) The absolute maximum of a function ...

- ☐ ... is always a relative maximum.
- ☐ ... can be a relative minimum.
- ☐ ... can be a relative maximum.
- ☐ ... always exists.

Answers

15.1 a) $f(x) = x^2 - 4$

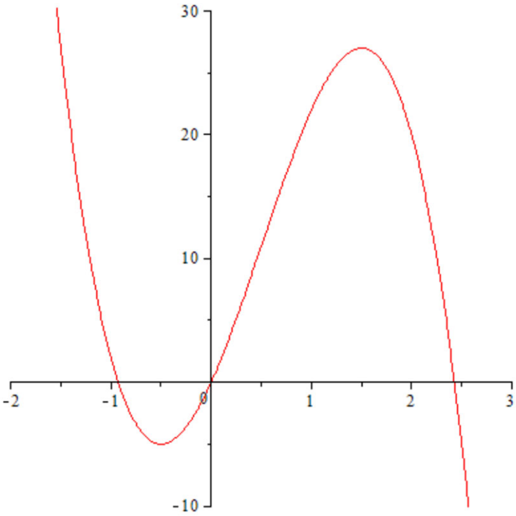


$f'(x) = 2x$
$f''(x) = 2$
$f'''(x) = 0$

i) $f'(x) = 0$ at $x_1 = 0$
$f''(x_1) = 2 > 0$ ⇒ relative minimum at $x_1 = 0$
no relative maximum

ii) $f''(x) = 2 \neq 0$ for all x ⇒ no point of inflection

b) $f(x) = -8x^3 + 12x^2 + 18x$



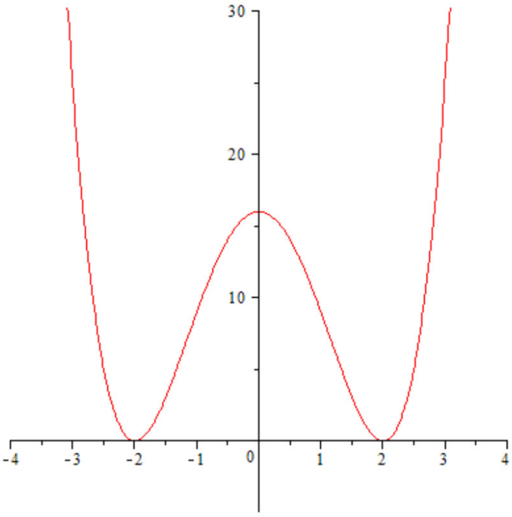
$f'(x) = -24x^2 + 24x + 18$
$f''(x) = -48x + 24$
$f'''(x) = -48$

i) $f'(x) = 0$ at $x_1 = -\frac{1}{2}$ and $x_2 = \frac{3}{2}$
$f''(x_1) = 48 > 0$ ⇒ relative minimum at $x_1 = -\frac{1}{2}$
$f''(x_2) = -48 < 0$ ⇒ relative maximum at $x_2 = \frac{3}{2}$

ii) $f''(x) = 0$ at $x_3 = \frac{1}{2}$

$f'''(x_3) = -48 \neq 0$ ⇒ point of inflection at $x_3 = \frac{1}{2}$

c) $s(t) = t^4 - 8t^2 + 16$



$s'(t) = 4t^3 - 16t$
$s''(t) = 12t^2 - 16$
$s'''(t) = 24t$

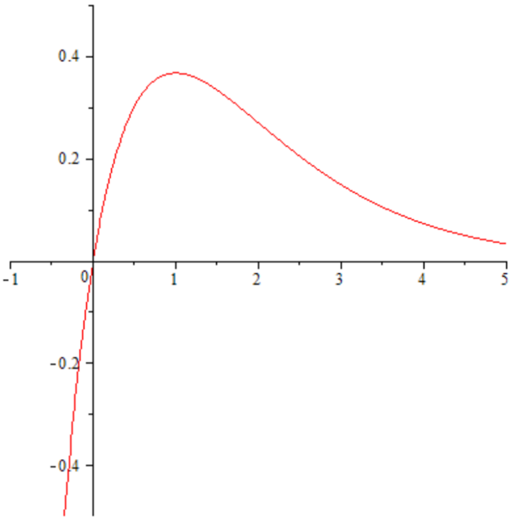
i) $s'(t) = 0$ at $t_1 = 0$, $t_2 = -2$, and $t_3 = 2$

$s''(t_1) = -16 < 0$ ⇒ relative maximum at $t_1 = 0$
$s''(t_2) = 32 > 0$ ⇒ relative minimum at $t_2 = -2$
$s''(t_3) = 32 > 0$ ⇒ relative minimum at $t_3 = 2$

ii) $s''(t) = 0$ at $t_4 = -\frac{2}{\sqrt{3}}$ and $t_5 = \frac{2}{\sqrt{3}}$

$s'''(t_4) = -\frac{48}{\sqrt{3}} \neq 0$ ⇒ point of inflection at $t_4 = -\frac{2}{\sqrt{3}}$
$s'''(t_5) = \frac{48}{\sqrt{3}} \neq 0$ ⇒ point of inflection at $t_5 = \frac{2}{\sqrt{3}}$

d) $f(x) = x\,e^{-x}$

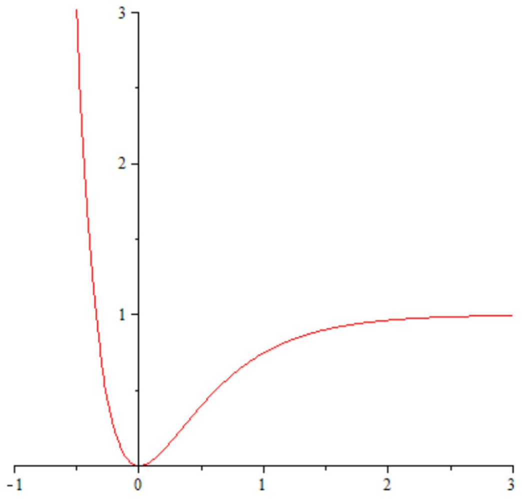


$f'(x) = e^{-x} - x\,e^{-x} = (1 - x)\,e^{-x}$
$f''(x) = -e^{-x} - (1 - x)\,e^{-x} = (x - 2)\,e^{-x}$
$f'''(x) = e^{-x} - (x - 2)\,e^{-x} = (3 - x)\,e^{-x}$

i) $f'(x) = 0$ at $x_1 = 1$
$f''(x_1) = -\frac{1}{e} < 0$ ⇒ relative maximum at $x_1 = 1$
no relative minimum

ii) $f''(x) = 0$ at $x_2 = 2$
$f'''(x_2) = \frac{1}{e^2} \neq 0$ ⇒ point of inflection at $x_2 = 2$

e) * $f(x) = (1 - e^{-2x})^2 = 1 - 2\,e^{-2x} + e^{-4x}$



$f'(x) = 4\,(e^{-2x} - e^{-4x})$
$f''(x) = 8\,(-\,e^{-2x} + 2\,e^{-4x})$
$f'''(x) = 16\,(e^{-2x} - 4\,e^{-4x})$

i) $f'(x) = 0$ at $x_1 = 0$
$f''(x_1) = 8 > 0$ ⇒ relative minimum at $x_1 = 0$
no relative maximum

ii) $f''(x) = 0$ at $x_2 = \frac{\ln(2)}{2} = 0.34\ldots$
$f'''(x_2) = -\,8 \neq 0$ ⇒ point of inflection at $x_2 = 0.34\ldots$

f) * $V'(r) = -\,D\left(-\frac{2a}{r^2} + \frac{2a^2}{r^3}\right) = \frac{2aD}{r^2}\left(1 - \frac{a}{r}\right)$
$V''(r) = -\,D\left(\frac{4a}{r^3} - \frac{6a^2}{r^4}\right) = \frac{2aD}{r^3}\left(\frac{3a}{r} - 2\right)$
$V'''(r) = -\,D\left(-\frac{12a}{r^4} + \frac{24a^2}{r^5}\right) = \frac{12aD}{r^4}\left(1 - \frac{2a}{r}\right)$

i) $V'(r) = 0$ at $r_1 = a$
$V''(r_1) = \frac{2D}{a^2} > 0$ ⇒ relative minimum at $r_1 = a$
no relative maximum

ii) $V''(r) = 0$ at $r_2 = \frac{3a}{2}$
$V'''(r_2) = -\,\frac{64D}{81a^3} \neq 0$ ⇒ point of inflection at $r_2 = \frac{3a}{2}$

15.2 **Relative** maximum at $x_1 = \frac{100}{3}$ → 33 or 34
P(33) = 51'843 CHF
P(34) = 51'816 CHF
$P(x) < P(x_1)$ if $x \neq x_1$ as there is no relative minimum
⇒ P = 51'843 CHF is the **absolute** maximum profit at x = 33.

15.3 (see next page)

15.3 $\overline{C}(x) = \frac{C(x)}{x} = \frac{1}{4}x + 4 + \frac{100}{x}$
$\overline{C}(x)$ has a **relative** minimum at $x_1 = 20$
$\overline{C}(20) = 14$ CHF
$\overline{C}(x) > \overline{C}(x_1)$ if $x \neq x_1$ as there is no relative maximum
$\Rightarrow$ $\overline{C} = 14$ CHF is the **absolute** minimum average cost at $x = 20$.

15.4 P(x) has a **relative** maximum at $x_1 = 15$ and a **relative** minimum at $x_2 = 20$.
$P(x_1) = 20'250$ CHF
$P(x) < P(x_1)$ if $x < x_1$ as there is no relative minimum on the interval $x < x_1$
$P(30) = 27'000$ CHF > 20'250 CHF (!)
$\Rightarrow$ P = 27'000 CHF is the **absolute** maximum profit at the endpoint $x = 30$.

15.5 P(x) has a point of inflection at $x_1 = 10$
$P(10) = 200$
$\Rightarrow$ point of inflection (10|200), i.e. when $x = 10$ (in the year 2020) and P = 200'000 CHF

15.6 a) 4^{th} statement
b) 3^{rd} statement
c) 3^{rd} statement