Exercises 14 — Differentiation rules
Coefficient/sum/product rule, higher-order derivatives

Objectives

- be able to apply the coefficient, sum, product rule to determine the derivative of a function.
- be able to determine a higher-order derivative of a function.

Problems

14.1 Determine the derivative by applying the **coefficient rule**:

a)	$f(x) = 3x^5$	b)	$f(x) = -4x^3$	c)	$f(x) = -x^{10}$
d)	$f(x) = a\cdot x^3$	e)	$f(x) = n\cdot x^{n-1}$	f)	$f(x) = 9\cdot 3^x$
g)	$s(t) = \frac{1}{2}g\cdot t^2$	h)	$S(T) = \alpha\cdot T^4$	i)	$C(x) = (-3x)^3$

14.2 Determine the derivative by applying the **sum rule**:

a)	$f(x) = x^5 + x^6$	b)	$f(x) = x^{10} - x^9$	c)	$f(x) = 1 + x + 3x^3$
d)	$f(x) = \frac{1}{4}x^4 + 3x^2 - 2$	e)	$f(x) = 3x^2(x - 2)$	f)	$f(x) = - 3x^8 + x^5 - 3x + 99$
g)	$f(x) = ax^2 + bx + c$	h)	$f(x) = 3(a^2 - 2ax + x^2)$	i)	$f(x) = \frac{x^3}{3} - \frac{3}{x^3}$
j)	$s(t) = s_0 + v_0t + \frac{1}{2}g\cdot t^2$	k)	$V(r) = -\frac{a}{r} + \frac{b}{r^2}$	l)	$C(n) = C_0(1 + nr)$

14.3 Determine the derivative by applying the **product rule**:

a)	$f(x) = x\cdot e^x$	b)	$f(x) = x^3\cdot 3^x$
c)	$f(x) = - 2x^5(x - 1)$	d)	$f(x) = (2x - 1)\cdot e^x$
e)	$f(x) = (2x - 1)(- 3x^2 - x + 1)$	f)	$f(x) = 3(1 - x^2)(x^{10} - x^9)$
g)	$V(r) = e^r\left(a\cdot r^2 - \frac{b}{r^3}\right)$	h)	$T(V) = \frac{1}{n\cdot R}\left(p + \frac{a\cdot n^2}{V^2}\right)(V - n\cdot b)$

14.4 Determine the derivative of the exponential functions below:

a)	$f(x) = e^{4x}$	b)	$f(x) = e^{-x}$
c)	$f(x) = e^{1 - \frac{x}{2}}$	d)	$f(x) = e^{-x^2}$
e)	$f(x) = e^{x^2-2x+5}$		

14.5 Determine the derivative by applying the appropriate differentiation rule(s), and simplify the expression as far as possible:

a)	$f(x) = (x - 2)\,e^{2x}$	b)	$f(x) = (2 - x^2)\,e^{-x}$
c)	$f(x) = (3x^3 - 2x^2 + x - 1)\,e^{-2x}$	d)	$f(x) = x^2\,e^{-x^2 - 2x}$
e)	$f(x) = ax\,e^{-\frac{x^2}{2}}$	f)	$P(v) = av^2\,e^{-bv^2}$

14.6 (see next page)

14.6 Determine the derivative (rate of change) of the indicated function at the indicated position:

a) f in 14.1 b) $x = 2$ b) s in 14.1 g) $t = 4$

c) f in 14.2 g) $x = -1$ d) f in 14.5 e) $x = 0$

14.7 Determine the second and third derivatives of the functions in problem …

a) … 14.1 a) b) … 14.2 g)

c) … 14.3 a) d) … 14.4 d)

e) … 14.5 b) f) … 14.5 e)

14.8 Determine the indicated higher-order derivatives:

a) $f''(-1)$ with function f in 14.1 a)

Hint:
- You have already determined $f''(x)$ in 14.7 a).

b) $f'''(2)$ with function f in 14.5 e)

Hint:
- You have already determined $f'''(x)$ in 14.7 f).

14.9 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) The third derivative of a function is a ...

- ☐ ... constant function if the second derivative is a quadratic function.
- ☐ ... quadratic function if the second derivative is a linear function.
- ☐ ... linear function if the first derivative is a quadratic function.
- ☐ ... constant function if the first derivative is a quadratic function.

b) The derivative of a ...

- ☐ ... product is the product of the derivatives of the single factors.
- ☐ ... product is the sum of the derivatives of the single factors.
- ☐ ... sum is the sum of the derivatives of the single addends.
- ☐ ... constant is the constant itself.

c) If $f(x) = c \cdot g(x) \cdot h(x)$ then $f'(x) = ...$

- ☐ ... 0
- ☐ ... $c \cdot g'(x) \cdot h'(x)$
- ☐ ... $c \cdot g(x) \cdot h'(x) + c \cdot g'(x) \cdot h(x)$
- ☐ ... $c \cdot g'(x) \cdot h'(x) + c \cdot g(x) \cdot h(x)$

Answers

14.1 a) $f'(x) = 3 \cdot 5x^4 = 15x^4$

b) $f'(x) = (-4)\ 3x^2 = -12x^2$

c) $f'(x) = (-1)\ 10x^9 = -10x^9$

d) $f'(x) = a \cdot 3x^2 = 3ax^2$

Hint:
- a is a constant.

e) $f'(x) = n(n-1)x^{n-2}$

f) $f'(x) = 9 \cdot 3^x \cdot \ln(3)$

g) $s'(t) = \frac{g}{2}\ 2t = gt$

Hints:
- The name of the function is s, and the variable is t.
- g is a constant.

h) $S'(T) = \alpha \cdot 4T^3 = 4\alpha T^3$

i) $C'(x) = -\ 81x^2$

14.2 a) $f'(x) = 5x^4 + 6x^5$ b) $f'(x) = 10x^9 - 9x^8$ c) $f'(x) = 1 + 9x^2$

d) $f'(x) = x^3 + 6x$ e) $f'(x) = 9x^2 - 12x$ f) $f'(x) = -\ 24x^7 + 5x^4 - 3$

g) $f'(x) = 2ax + b$ h) $f'(x) = -\ 6a + 6x$ i) $f'(x) = x^2 + \frac{9}{x^4}$

j) $s'(t) = v_0 + gt$ k) $V'(r) = \frac{a}{r^2} - \frac{2b}{r^3}$ l) $C'(n) = C_0 \cdot r$

14.3 a) $f'(x) = e^x + x \cdot e^x$

b) $f'(x) = 3x^2 \cdot 3^x + x^3 \cdot 3^x \cdot \ln(3)$

c) $f'(x) = -\ 2(5x^4(x - 1) + x^5)$

d) $f'(x) = 2 \cdot e^x + (2x - 1) \cdot e^x$

e) $f'(x) = 2(-\ 3x^2 - x + 1) + (2x - 1)\ (-\ 6x - 1)$

f) $f'(x) = 3(-\ 2x(x^{10} - x^9) + (1 - x^2)\ (10x^9 - 9x^8))$

g) $V'(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3}\right) + e^r \left(2a \cdot r + \frac{3b}{r^4}\right)$

Hints:
- V is the name of the function, and r is the variable.
- a and b are constants.

h) $T'(V) = \frac{1}{n \cdot R}\left(-\frac{2a \cdot n^2}{V^3}(V - n \cdot b) + \left(p + \frac{a \cdot n^2}{V^2}\right)\right)$

Hints:
- T is the name of the function, and V is the variable.
- n, R, p, a, and b are constants.

14.4 a) $f'(x) = 4\ e^{4x}$ b) $f'(x) = (-1)\ e^{-x} = -\ e^{-x}$

c) $f'(x) = -\frac{1}{2}\, e^{1 - \frac{x}{2}}$ d) $f'(x) = -\ 2x \cdot e^{-x^2}$

e) $f'(x) = (2x - 2)\ e^{x^2 - 2x + 5}$

14.5 a) $f'(x) = e^{2x} + (x - 2)\, 2\, e^{2x} = (2x - 3)\, e^{2x}$

b) $f'(x) = -2x\, e^{-x} + (2 - x^2)(-1)\, e^{-x} = (x^2 - 2x - 2)\, e^{-x}$

c) $f'(x) = (9x^2 - 4x + 1)\, e^{-2x} + (3x^3 - 2x^2 + x - 1)(-2)\, e^{-2x} = (-6x^3 + 13x^2 - 6x + 3)\, e^{-2x}$

d) $f'(x) = 2x\, e^{-x^2-2x} + x^2(-2x - 2)\, e^{-x^2-2x} = 2x\,(1 - x - x^2)\, e^{-x^2-2x}$

e) $f'(x) = a\left(e^{-\frac{x^2}{2}} + x\,(-x)\, e^{-\frac{x^2}{2}}\right) = a\,(1 - x^2)\, e^{-\frac{x^2}{2}}$

f) $P'(v) = a\left(2v\, e^{-bv^2} + v^2(-2bv)\, e^{-bv^2}\right) = 2av\,(1 - bv^2)\, e^{-bv^2}$

14.6 a) $f'(2) = -48$ b) $s'(4) = 4g$

c) $f'(-1) = -2a + b$ d) $f'(0) = a$

14.7 a) 14.1 a)
$f''(x) = 15\cdot 4x^3 = 60x^3$
$f'''(x) = 60\cdot 3x^2 = 180x^2$

b) 14.2 g)
$f''(x) = 2a\cdot 1 = 2a$
$f'''(x) = 0$

c) 14.3 a)
$f''(x) = e^x + (e^x + x\cdot e^x) = (x + 2)\, e^x$
$f'''(x) = e^x + (x + 2)\, e^x = (x + 3)\, e^x$

d) 14.4 d)
$f''(x) = -2\left(e^{-x^2} + x\,(-2x)\, e^{-x^2}\right) = 2\,(2x^2 - 1)\, e^{-x^2}$
$f'''(x) = 2\left(4x\, e^{-x^2} + (2x^2 - 1)(-2x)\, e^{-x^2}\right) = 4x\,(-2x^2 + 3)\, e^{-x^2}$

e) 14.5 b)
$f''(x) = (2x - 2)\, e^{-x} + (x^2 - 2x - 2)(-1)\, e^{-x} = (4x - x^2)\, e^{-x}$
$f'''(x) = (4 - 2x)\, e^{-x} + (4x - x^2)(-1)\, e^{-x} = (x^2 - 6x + 4)\, e^{-x}$

f) 14.5 e)
$f''(x) = a\left(-2x\, e^{-\frac{x^2}{2}} + (1 - x^2)(-x)\, e^{-\frac{x^2}{2}}\right) = a\,(x^3 - 3x)\, e^{-\frac{x^2}{2}}$
$f'''(x) = a\left((3x^2 - 3)\, e^{-\frac{x^2}{2}} + (x^3 - 3x)(-x)\, e^{-\frac{x^2}{2}}\right) = a\,(-x^4 + 6x^2 - 3)\, e^{-\frac{x^2}{2}}$

14.8 a) $f''(-1) = -60$

b) $f'''(2) = a(-16 + 6\cdot 4 - 3)\, e^{-\frac{4}{2}} = \frac{5a}{e^2}$

14.9 a) 4th statement

b) 3rd statement

c) 3rd statement