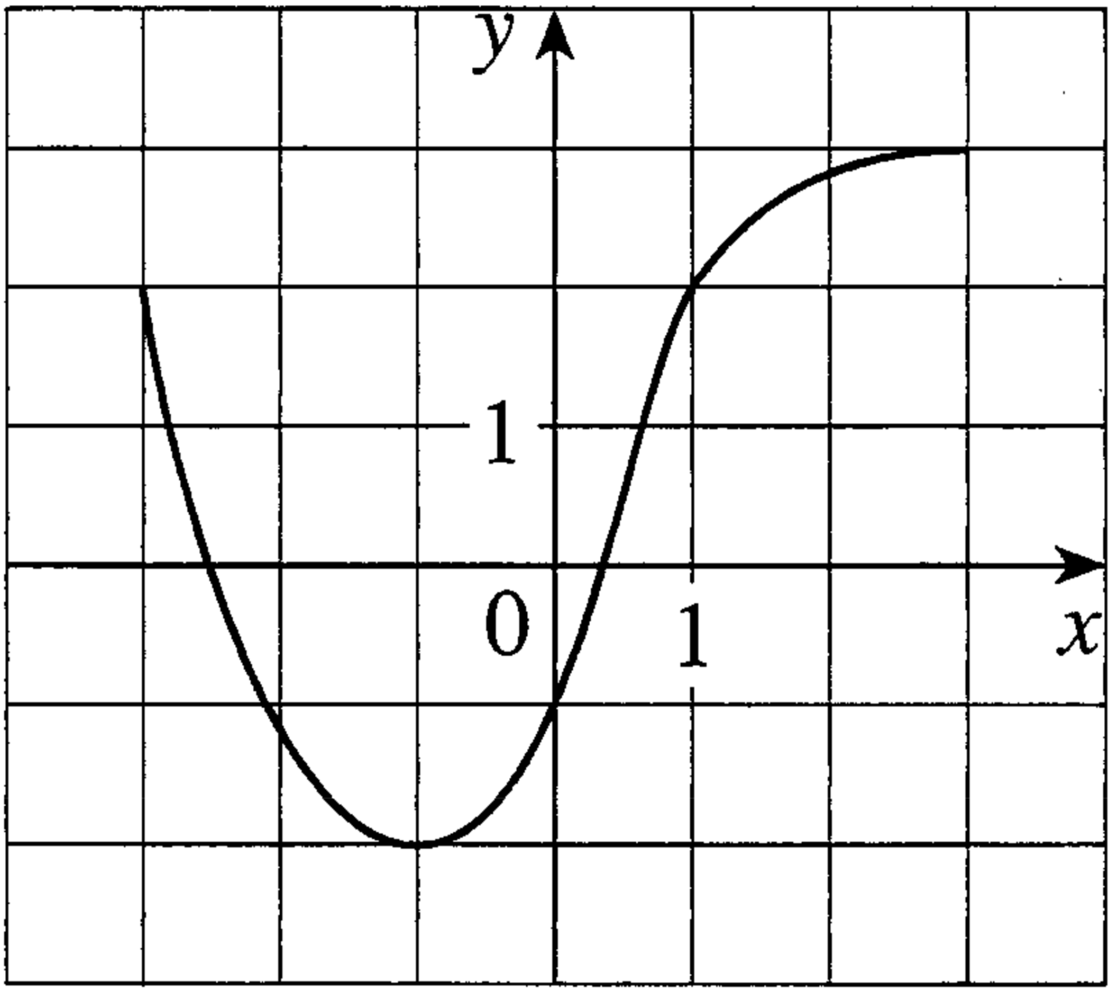
Exercises 13 Derivative
Derivative (rate of change), derivative (derived function) of constant/power/exponential functions

Objectives

- be able to estimate a derivative (rate of change) out of the graph of a function.
- be able to state the derivative (rate of change) of a constant/linear function.
- be able to determine the derivative (derived function) of a constant/linear function.
- be able to determine the derivative (derived function) of a basic power/exponential function.
- be able to determine a derivative (rate of change) of a basic power/exponential function.

Problems

13.1 The graph of a function f ist given as follows:



Estimate the derivative (rate of change) $f'(x_0)$ at the given position x_0:

a) $x_0 = -1$ b) $x_0 = 0$

c) $x_0 = 1$ d) $x_0 = -2$

Hints:
- Draw the tangent to the graph of f at the given position x_0.
- Estimate the slope of the tangent.

13.2 The graph of a constant or linear function is a straight line. Therefore, the "tangent" to the graph through any point of the graph is the graph itself.

For each of the following functions $f: \mathbb{R} \to \mathbb{R}, x \mapsto y = f(x) = ...$

i) ... draw the graph of f.

ii) ... state the derivative (rate of change) $f'(x_0)$ at the given position x_0.

a) $f(x) = 3$ $x_0 = 2$

b) $f(x) = c$ $(c \in \mathbb{R})$ any $x_0 \in \mathbb{R}$

c) $f(x) = 2x - 3$ $x_0 = 4$

d) $f(x) = mx + q$ $(m \in \mathbb{R} \setminus \{0\}, q \in \mathbb{R})$ any $x_0 \in \mathbb{R}$

e) * $f(x) = |x|$ any $x_0 \in \mathbb{R}$

13.3 Determine $f'(x)$:

a) $f(x) = 3$ b) $f(x) = 0$ c) $f(x) = -1$

d) $f(x) = x^3$ e) $f(x) = x^4$ f) $f(x) = x^5$

g) $f(x) = x^{17}$ h) $f(x) = x^{200}$ i) $f(x) = x^{100'001}$

j) $f(x) = x^{-1}$ k) $f(x) = x^{-2}$ l) $f(x) = x^{-17}$

m) $f(x) = \frac{1}{x}$ n) $f(x) = \frac{1}{x^3}$ o) $f(x) = \frac{1}{x^{99}}$

13.4 Determine $f'(x)$:

a) $f(x) = 3^x$ b) $f(x) = 5^x$ c) $f(x) = 18^x$

d) $f(x) = \left(\frac{2}{3}\right)^x$ e) $f(x) = \left(\frac{13}{17}\right)^x$ f) $f(x) = \left(\frac{1}{4}\right)^x$

g) $f(x) = \left(\frac{1}{e}\right)^x$ h) * $f(x) = \left(\frac{3}{e}\right)^x$ i) * $f(x) = \left(\frac{e}{3}\right)^x$

13.5 Determine the derivative (rate of change) $f'(x_0)$ of the function f at the indicated position x_0:

a) $f(x) = x$

i) $x_0 = 0$ ii) $x_0 = 1$ iii) $x_0 = -2$

b) $f(x) = x^5$

i) $x_0 = 0$ ii) $x_0 = 2$ iii) $x_0 = -\frac{2}{3}$

c) $f(x) = x^{-4}$

i) $x_0 = -1$ ii) $x_0 = -\frac{4}{3}$ iii) $x_0 = 0$

d) $f(x) = \left(\frac{2}{3}\right)^x$

i) $x_0 = 0$ ii) $x_0 = 1$ iii) $x_0 = -2$

13.6 * (see next page)

13.6 * The derivative (rate of change) $f'(x_0)$ of f at the position x_0 can be determined by looking at the secant through the points $A(x_0 \mid f(x_0))$ and $B(x_0+\Delta x \mid f(x_0+\Delta x))$ of the graph of f. The slope of this secant tends towards the slope of the tangent through $A(x_0 \mid f(x_0))$ as Δx tends towards 0.

It has been shown in class how to determine $f'(x_0)$ for the quadratic function $f(x) = x^2$.

Find $f'(x_0)$ for the following functions f:

a) $f(x) = x^3$ b) $f(x) = \frac{1}{x^2}$

13.7 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) The derivative (rate of change) of a function f at the position x_0 is a ...

- ☐ ... real number.
- ☐ ... function.
- ☐ ... tangent.
- ☐ ... graph.

b) The derivative (derived function) f ' of a function f is a ...

- ☐ ... real number.
- ☐ ... function.
- ☐ ... tangent.
- ☐ ... graph.

c) $f'(x_0)$ is the slope of the ...

- ☐ ... secant through the points $(0|0)$ and $(x_0|f(x_0))$.
- ☐ ... secant through the points $(x_0+\Delta x|f(x_0+\Delta x))$ and $(x_0|f(x_0))$.
- ☐ ... tangent to the graph of f through $(x_0|f(x_0))$.
- ☐ ... tangent to the graph of f ' through $(x_0|f(x_0))$.

Answers

13.1 a) $f'(-1) \approx 0$ b) $f'(0) \approx 2$
c) $f'(1) \approx \frac{3}{2}$ d) $f'(-2) \approx -\frac{5}{3}$

13.2 a) i) ...
ii) $f'(2) = 0$
b) i) ...
ii) $f'(x_0) = 0$
c) i) ...
ii) $f'(4) = 2$
d) i) ...
ii) $f'(x_0) = m$
e) * i) ...
ii) $f'(x_0) = \begin{cases} 1 & (x_0 > 0) \\ -1 & (x_0 < 0) \\ \text{not defined} & (x_0 = 0) \end{cases}$

13.3
a) $f'(x) = 0$ b) $f'(x) = 0$ c) $f'(x) = 0$
d) $f'(x) = 3x^2$ e) $f'(x) = 4x^3$ f) $f'(x) = 5x^4$
g) $f'(x) = 17x^{16}$ h) $f'(x) = 200x^{199}$ i) $f'(x) = 100'001x^{100'000}$
j) $f'(x) = -x^{-2}$ k) $f'(x) = -2x^{-3}$ l) $f'(x) = -17x^{-18}$
m) $f'(x) = -\frac{1}{x^2}$ n) $f'(x) = -\frac{3}{x^4}$ o) $f'(x) = -\frac{99}{x^{100}}$

13.4
a) $f'(x) = 3^x \ln(3)$ b) $f'(x) = 5^x \ln(5)$ c) $f'(x) = 18^x \ln(18)$
d) $f'(x) = \left(\frac{2}{3}\right)^x \ln\left(\frac{2}{3}\right)$ e) $f'(x) = \left(\frac{13}{17}\right)^x \ln\left(\frac{13}{17}\right)$
f) $f'(x) = \left(\frac{1}{4}\right)^x \ln\left(\frac{1}{4}\right) = -\frac{\ln(4)}{4^x}$

Hint:
- Logarithm rules (see formulary) can be applied in order to simplify the result.

g) $f'(x) = -\frac{1}{e^x}$ h) * $f'(x) = \left(\frac{3}{e}\right)^x (\ln(3) - 1)$ i) * $f'(x) = \left(\frac{e}{3}\right)^x (1 - \ln(3))$

13.5 a) $f'(x) = 1$
i) $f'(0) = 1$ ii) $f'(1) = 1$ iii) $f'(-2) = 1$
b) $f'(x) = 5x^4$
i) $f'(0) = 0$ ii) $f'(2) = 80$ iii) $f'\left(-\frac{2}{3}\right) = \frac{80}{81}$
c) $f'(x) = -\frac{4}{x^5}$
i) $f'(-1) = 4$ ii) $f'\left(-\frac{4}{3}\right) = \frac{243}{256}$ iii) $f'(0)$ is not defined
d) (see next page)

d) $f'(x) = \left(\frac{2}{3}\right)^x \ln\left(\frac{2}{3}\right)$

i) $f'(0) = \ln\left(\frac{2}{3}\right)$ ii) $f'(1) = \frac{2}{3}\ln\left(\frac{2}{3}\right)$ iii) $f'(-2) = \frac{9}{4}\ln\left(\frac{2}{3}\right)$

13.6 * a) $f'(x_0) = 3x_0^2$ b) $f'(x_0) = -\frac{2}{x_0^3}$

13.7 a) 1st statement

b) 2nd statement

c) 3rd statement