Exercises 4 Linear function and equations
Linear function, simple interest, cost, revenue, profit, break-even

Objectives

- be able to think of a relation between two quantities as a function.
- be able to determine the domain, the codomain, the range of a given function.
- be able to draw the graph of a given linear function.
- be able to determine slope and intercept of a linear function.
- know some examples of linear functions in economic and everyday life applications.
- know and understand the term "simple interest".
- be able to perform simple interest calculation.
- know and understand the terms "fixed costs", "variable costs", "total cost", "total revenue", "total profit" and "break-even value".
- be able to apply the concept of linear functions to a new problem.

Problems

4.1 A taxi driver charges the following fare:

8.00 CHF plus 1.50 CHF per kilometre

Think of the taxi fare as a function f.

a) Determine the domain D, the codomain B, and the range E of the function.

b) Draw the graph of the function f.

4.2 The taxi fare as described in problem 4.1 can be thought of as a linear function which assigns a fare y (in CHF) to each distance x (in km):

$$\begin{aligned} f:\ & \mathbb{R}^+ \to \mathbb{R}^+ \\ & x \mapsto y = f(x) = ax + b \end{aligned}$$

Determine the values of a and b.

4.3 Find at least two more examples of linear functions in economics or in an everyday life context.

4.4 Graph the linear functions below, and state both slope and intercept:

a) $$\begin{aligned} f:\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto y = f(x) = -2 \end{aligned}$$

b) $$\begin{aligned} f:\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto y = f(x) = 3x - 4 \end{aligned}$$

c) $$\begin{aligned} f:\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto y = f(x) = -x + 3 \end{aligned}$$

4.5 Simple interest at a rate of 0.5% is paid on an initial bank balance of 5000 CHF.

a) Determine the interest that is paid each year.

b) Determine the balance after ten years' time.

c) Determine both slope and intercept of the corresponding linear function.

4.6 In general, if an initial capital C_0 pays simple interest at an annual rate r (e.g. r = 1.5% = 0.015), the capital C_n after n years is given by the formula below (see formulary):

$$C_n = C_0\,(1 + nr)$$

a) Verify that the given formula is correct.

b) Determine both slope and intercept of the corresponding linear function.

4.7 An initial capital C_0 = 1200 CHF pays simple interest at an annual interest rate of 1.5%.

a) After how many years will the capital exceed 2000 CHF?

b) At what annual interest rate (rounded to 0.05%) would the capital exceed 2000 CHF after 20 years' time?

Hint:
- Use the formula given in problem 4.6 and solve it for n and r respectively.

4.8 A satellite phone company offers three different tariffs:

Tariff A:	monthly basic fee of 10 CHF plus 0.20 CHF per minute
Tariff B:	monthly basic fee of 25 CHF plus 0.10 CHF per minute
Tariff C:	no basic fee, 0.60 CHF per minute

Think of the the three tariffs as linear functions.

a) Draw the graphs of the three functions in one common coordinate system.

b) Determine the total fee for each tariff for a monthly phone call duration of 1 hour.

c) For what monthly phone call duration tariff A is cheaper than tariff C?

d) For what monthly phone call duration tariff B is cheaper than tariff A?

4.9 (from: Bittinger, Ellenbogen: Calculus and its applications, Pearson 2007, ISBN 0-321-48543-2)

EXAMPLE 9 Business: Total Cost. Raggs, Ltd., a clothing firm, has **fixed costs** of \$10,000 per year. These costs, such as rent, maintenance, and so on, must be paid no matter how much the company produces. To produce x units of a certain kind of suit, it costs \$20 per suit (unit) in addition to the fixed costs. That is, the **variable costs** for producing x of these suits are $20x$ dollars. These costs are due to the amount produced and stem from items such as material, wages, fuel, and so on. The **total cost** $C(x)$ of producing x suits in a year is given by a function C:

$$C(x) = (\text{Variable costs}) + (\text{Fixed costs}) = 20x + 10{,}000.$$

a) Graph the variable-cost, the fixed-cost, and the total-cost functions.

b) What is the total cost of producing 100 suits? 400 suits?

4.10 (see next page)

4.10 (from: Bittinger, Ellenbogen: Calculus and its applications, Pearson 2007, ISBN 0-321-48543-2)

EXAMPLE 10 Business: Profit-and-Loss Analysis. When a business sells an item, it receives the *price* paid by the consumer (this is normally greater than the *cost* to the business of producing the item).

a) The **total revenue** that a business receives is the product of the number of items sold and the price paid per item. Thus, if Raggs, Ltd., sells x suits at \$80 per suit, the total revenue $R(x)$, in dollars, is given by

$$R(x) = \text{Unit price} \cdot \text{Quantity sold} = 80x.$$

If $C(x) = 20x + 10{,}000$ (see Example 9), graph R and C using the same set of axes.

b) The **total profit** that a business receives is the amount left after all costs have been subtracted from the total revenue. Thus, if $P(x)$ represents the total profit when x items are produced and sold, we have

$$P(x) = (\text{Total revenue}) - (\text{Total costs}) = R(x) - C(x).$$

Determine $P(x)$ and draw its graph using the same set of axes as was used for the graph in part (a).

c) The company will *break even* at that value of x for which $P(x) = 0$ (that is, no profit and no loss). This is the point at which $R(x) = C(x)$. Find the **break-even value** of x.

4.11 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a)
- ☐ Each straight line in a coordinate system can be considered as the graph of a linear function.
- ☐ The graph of each linear function is a straight line.
- ☐ If y is proportional to x, x is not necessarily proportional to y.
- ☐ The range of each linear function is ℝ.

b) f cannot be a linear function if ...
- ☐ ... the graph of f is a straight line.
- ☐ ... $f(x) \neq x$ for at least one element x of the domain of f.
- ☐ ... the domain of f does not consist of all real numbers.
- ☐ ... $f(x) = ax + b$ and a depends on x.

c) In a simple interest scheme ...
- ☐ ... the relation between time and capital does not correspond to a linear function.
- ☐ ... the interest paid at the end of each period depends on the capital at the end of the previous period.
- ☐ ... the interest paid at the end of each period is always the same amount of money.
- ☐ ... the capital doubles in less than 5 years if the annual interest rate is 20%.

Answers

4.1 a) $D = \mathbb{R}^+$ (distance/km)
$B = \mathbb{R}^+$ (fare/CHF)
$E = \{y: y \in \mathbb{R}^+ \text{ and } y > 8\}$ or $E = (8,\infty)$

b) ...

4.2 $a = 1.5$, $b = 8$

4.3 ...

4.4 a) Slope $a = 0$, intercept $b = -2$

b) Slope $a = 3$, intercept $b = -4$

c) Slope $a = -1$, intercept $b = 3$

4.5 a) 25 CHF

b) 5250 CHF

c) $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$
$x \mapsto y = f(x) = ax + b$

Slope $a = 25$, intercept $b = 5000$

4.6 a) Interest paid each year $= r \cdot C_0$
Capital C_n after n years $= C_0 + n \cdot (r \cdot C_0) = C_0\,(1 + nr)$

b) Slope $a = r \cdot C_0$, intercept $b = C_0$

Hints:
- Compare the formula $C_n = C_0\,(1 + nr)$ with the general form of the equation of a linear function.
- $C_n = C_0\,(1 + nr) = an + b = f(n)$

4.7 a) $n = \dfrac{\frac{C_n}{C_0} - 1}{r}$ where $C_0 = 1200$ CHF, $C_n = 2000$ CHF, $r = 1.5\% = 0.015$

$\Rightarrow\ n = 44.4... \ \to\ 45$ years

b) $r = \dfrac{\frac{C_n}{C_0} - 1}{n}$ where $C_0 = 1200$ CHF, $C_n = 2000$ CHF

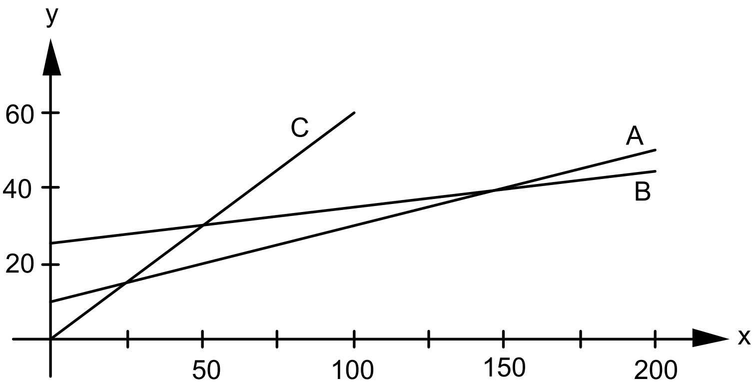
$n = 20 \ \Rightarrow\ r = 0.03333... = 3.333...\%$
$n = 19 \ \Rightarrow\ r = 0.03508... = 3.508...\%$

$\Rightarrow\ r \in \{3.35\%, 3.40\%, 3.45\%, 3.50\%\}$

4.8 a) Tariff A: $A\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$
$x \mapsto y = A(x) = 0.2\,x + 10$

Tariff B: $B\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$
$x \mapsto y = B(x) = 0.1\,x + 25$

Tariff C: $C\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$
$x \mapsto y = C(x) = 0.6\,x$
Direct proportionality: fee y is direct proportional to phone call duration x.



b) Tariff A: 22 CHF
Tariff B: 31 CHF
Tariff C: 36 CHF

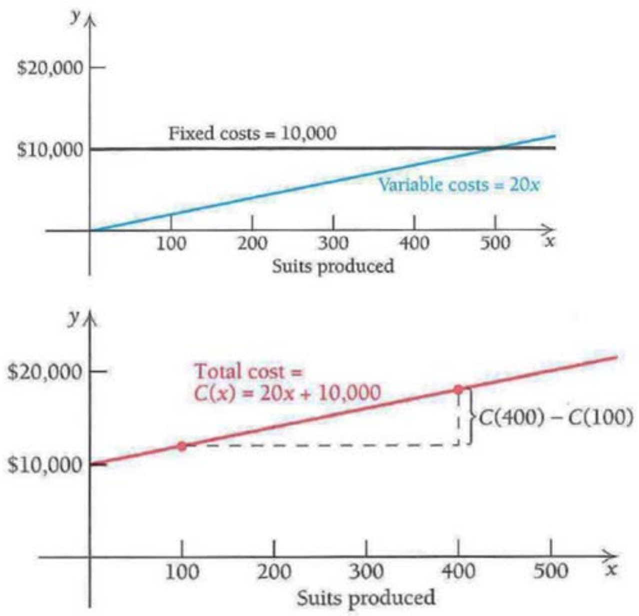
c) over 25 min

Hint:
- Solve the equation A(x) = C(x) for x.

d) over 150 min

4.9

a) The variable-cost and fixed-cost functions appear in the graph on the left below. The total-cost function is shown in the graph on the right. From a practical standpoint, the domains of these functions are nonnegative integers 0, 1, 2, 3, and so on, since it does not make sense to make either a negative number or a fractional number of suits. It is common practice to draw the graphs as though the domains were the entire set of nonnegative real numbers.



b) The total cost of producing 100 suits is

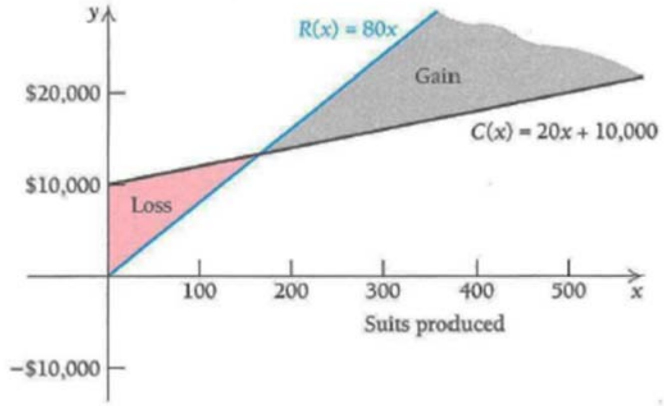
$$C(100) = 20 \cdot 100 + 10{,}000 = \$12{,}000.$$

The total cost of producing 400 suits is

$$\begin{aligned} C(400) &= 20 \cdot 400 + 10{,}000 \\ &= \$18{,}000. \end{aligned}$$

4.10

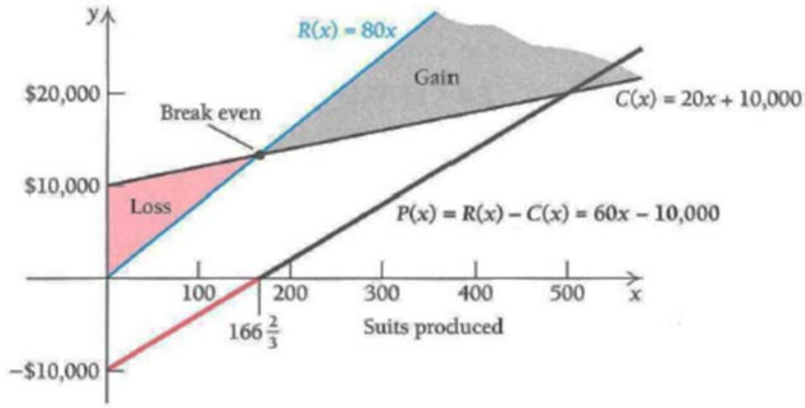
a) The graphs of $R(x) = 80x$ and $C(x) = 20x + 10{,}000$ are shown below. When $C(x)$ is above $R(x)$, a loss will occur. This is shown by the region shaded red. When $R(x)$ is above $C(x)$, a gain will occur. This is shown by the region shaded gray.



b) To find P, the profit function, we have

$$P(x) = R(x) - C(x) = 80x - (20x + 10{,}000)$$
$$= 60x - 10{,}000.$$

The graph of $P(x)$ is shown by the heavy line. The red portion of the line shows a "negative" profit, or loss. The black portion of the heavy line shows a "positive" profit, or gain.



c) To find the break-even value, we solve $R(x) = C(x)$:

$$R(x) = C(x)$$
$$80x = 20x + 10{,}000$$
$$60x = 10{,}000$$
$$x = 166\tfrac{2}{3}.$$

How do we interpret the fractional answer, since it is not possible to produce $\frac{2}{3}$ of a suit? We simply round to 167. Estimates of break-even values are usually sufficient since companies want to operate well away from break-even values in order to maximize profit.

4.11 a) 2nd statement

b) 4th statement

c) 3rd statement