Exercises 9 Exponential function and equations
Compound interest, exponential function

Objectives

- be able to calculate the future capital that is invested at an interest rate which is compounded annually.
- be able to treat compound interest tasks.
- be able to graph an exponential function out of its equation.
- be able to determine the equation of an exponential function out of the coordinates of two points of the graph.
- be able to treat applied tasks by means of an exponential function.

Problems

9.1 Compound interest at an annual rate r is paid on an initial capital C_0.

a) Assume an initial capital C_0 = 1000.00 CHF, and an annual interest rate r = 2%.
Determine the capital after one, two, three, four, and five years' time.

b) Try to develop a formula which allows you to calculate the capital C_n after n years' time for any values of C_0, r, and n.

9.2 What is the future capital if 8000 CHF is invested for 10 years at 12% compounded annually?

9.3 What present value amounts to 10'000 CHF if it is invested for 10 years at 6% compounded annually?

9.4 At what interest rate, compounded annually, would 10'000 CHF have to be invested to amount to 14'071 CHF in 7 years?

9.5 Ms Smith wants to invest 150'000 CHF for five years. Bank A offers an interest rate of 6.5% compounded annually. Bank B offers to pay 200'000 CHF after five years. Which bank makes the better offer?

9.6 The purchase of Alaska cost the United States $ 7 million in 1869. If this money had been placed in a savings account paying 6% compounded annually, how much money would be available from this investment in 2020?

9.7 Mary Stahley invested $2500 in a 36-month certificate of deposit (CD) that earned 8.5% annual simple interest. When the CD matured, she invested the full amount in a mutual fund that had an annual growth equivalent to 18% compounded annually. How much was the mutual fund worth after 9 years?

9.8 A capital is invested for 4 years at 4% and for 3 more years at 6%, compounded annually. Eventually, the capital amounts to 72'000 CHF.

a) Determine the initial capital.

b) What is the average interest rate with respect to the whole period of time?

9.9 An unknown initial capital is invested at an unknown interest rate, compounded annually. After 2 years, the capital amounts to 5'891.74 CHF, and after another 5 years the capital is 6'997.54 CHF.
Determine both initial capital and interest rate.

9.10 Look at the following exponential function:

$$\begin{aligned} f:\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto y = f(x) = 2^x \end{aligned}$$

a) Establish a table of values of f for the interval $-3 \le x \le 3$.

b) Draw the graph of f in the interval $-3 \le x \le 3$ into a Cartesian coordinate system.

9.11 Graph the following exponential functions into one coordinate system:

$$\begin{aligned} f_1:\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto y = f_1(x) = 2^x \end{aligned}$$

$$\begin{aligned} f_2:\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto y = f_2(x) = 0.2^x \end{aligned}$$

$$\begin{aligned} f_3:\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto y = f_3(x) = 3 \cdot 0.5^x \end{aligned}$$

$$\begin{aligned} f_4:\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto y = f_4(x) = -2 \cdot 3^x \end{aligned}$$

9.12 The graph of an exponential function contains the points P and Q.
Determine the equation of the exponential function.

a) $P(0|1.02)$ $Q(1|1.0302)$

b) $P(1|12)$ $Q(3|192)$

c) $P(0|10'000)$ $Q(5|777.6)$

d) $P(5|16)$ $Q\left(9|\frac{1}{16}\right)$

9.13 A house that 20 years ago was worth \$160'000 has increased in value by 4% each year because of inflation. What is its worth today?

9.14 Suppose a country has a population of 20 million and projects a growth rate of 2% per year for the next 20 years. What will the population of this country be in 10 years?

9.15 A ball is dropped from a height of 12.8 meters. It rebounds 3/4 of the height from which it falls every time it hits the ground. How high will the ball bounce after it strikes the ground for the forth time?

9.16 A machine is valued at \$10'000. The depreciation at the end of each year is 20% of its value at the beginning of the year. Find its value at the end of 4 years.

9.17 The size of a certain bacteria culture grows exponentially. At 8 a.m. and 11 a.m. the number of bacteria was 2'300 and 18'400, respectively. Determine the number of bacteria at 1.30 p.m.

9.18 In a physical experiment the number of radioactive nuclei in a certain preparation decreases exponentially. 5 hours after the start of the experiment $1.56 \cdot 10^{16}$ nuclei were counted. 3 hours later, the number has fallen to $3.05 \cdot 10^{13}$. What was the number of nuclei at the beginning of the experiment?

9.19 A capital pays interest, compounded annually. What is the interest rate such that the capital doubles in 20 years?

9.20 * Suppose that the number y of otters t years after they were reintroduced into a wild and scenic river is given by

$$y = 2500 - 2490 \cdot e^{-0.1 \cdot t}$$

a) Find the population when the otters were introduced.

b) Draw the graph of the function f: $t \rightarrow y = f(t)$.

c) What is the expected upper limit of the number of otters?

9.21 * The consumer price index (CPI) is calculated by averaging the prices of various items after assigning a weight to each item. The following table gives the consumer price indexes for selected years from 1940 through 2002:

Year	**CPI**	**Year**	**CPI**
1940	14.0	1980	82.4
1950	24.1	1990	130.7
1960	29.6	2000	172.2
1970	38.8	2002	179.9

a) Find an equation that models these data, i.e. try to find the parameters a and c of the exponential function f: $x \mapsto y = f(x) = c \cdot a^x$ (x = years after 1900, y = CPI) that fits the data.

b) Use the model to predict the CPI in 2010.

9.22 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) In a compound interest scheme ...

- ☐ ... the graph that represents the growth of the capital is a parabola.
- ☐ ... the interest paid at the end of each period only depends on the interest rate.
- ☐ ... the interest rate depends on the capital of the previous period.
- ☐ ... the capital grows exponentially.

b) The graph of an exponential function ...

- ☐ ... is a parabola.
- ☐ ... is a straigh line if the initial value is equal to zero.
- ☐ ... never intersects the y-axis.
- ☐ ... never touches the x-axis.

c) If a quantity grows exponentially in time ...

- ☐ ... the growth factor itself grows.
- ☐ ... the growth factor depends on the initial value.
- ☐ ... the quantity doubles in one year if the annual growth factor is 100%.
- ☐ ... the quantity doubles in constant time intervals.

Answers

9.1 a) $C_0 = 1000.00$ CHF $C_1 = 1020.00$ CHF $C_2 = 1040.40$ CHF
$C_3 = 1061.21$ CHF $C_4 = 1082.43$ CHF $C_5 = 1104.08$ CHF

b) $C_n = C_0\,(1 + r)^n$

9.2 $C_{10} = 24'846.79$ CHF

9.3 $C_0 = 5'583.95$ CHF

9.4 $r = 5\%$

9.5 Bank A: $C_5 = 205'513.00$ CHF
Bank B: $C_5 = 200'000.00$ CHF

9.6 $C_{151} = \$\ 46'375$ million (rounded to millions)

9.7 \$13'916.24

2 periods: 3 years of simple interest, 9 years of compound interest

- 3 years of simple interest:
  $C_n = C_0(1 + nr)$ where $C_0 = \$2500$, $n = 3$, $r = 8.5\% = 0.085$
  $\Rightarrow\ C_3 = \$3137.50$
- 9 years of compound interest:
  $C_n = C_0\, q^n$ where $C_0 = ...$ (= C_3 after first 3 years), $q = 1 + 18\% = 1.18$, $n = 9$
  $\Rightarrow\ C_9 = \$13'916.24$

9.8 a) $C_0 = 51'675$ CHF

Hints:
- First, look at the second period (3 years, starting after 4 years from now), and calculate the capital at the beginning of this second period.
- Then, calculate the initial capital.

b) $r = 4.85\%$

Hint:
- The average interest rate r must be such that
  $C_n = C_0\, q^n$ where C_0 = initial capital, C_n = capital after the whole 7 years, $n = 7$, $q = 1 + r$

9.9 $r = 3.5\%$, $C_0 = 5'500.00$ CHF

Hints:
- First, look at the second period of 5 years, where $C_0 = 5'891.74$ CHF and $C_5 = 6'997.54$ CHF.
- The 5'891.74 CHF can be considered as the capital C_2 at the end of the first 2 years if C_0 is the initial capital at the very beginning of the 7 years.

9.10 ...

9.11 ...

9.12 a) $y = f(x) = 1.02 \cdot 1.01^x$

Hints:
- The equation of an exponential function is $y = f(x) = c \cdot a^x$
- If P(0|1.02) and Q(1|1.0302) are points of the graph of the exponential function, their coordinates must fulfil the equation of the exponential function, i.e. $1.02 = f(0) = c \cdot a^0$ and $1.0302 = f(1) = c \cdot a^1$
- Solve the two equations for c and a.

b) $y = f(x) = 3 \cdot 4^x$

c) $y = f(x) = 10'000 \cdot 0.6^x$

d) $y = f(x) = 16'384 \cdot 0.25^x$

9.13 $350'580 (rounded)

Hint:
- The relation between time t and the value V of the house is an exponential function:
$V = f(t) = V_0 \cdot a^t$
where V = value after time t, V_0 = initial value (at t = 0) = $160'000, a = growth factor = 1 + 4% = 1.04

9.14 24.4 million (rounded)

9.15 4.05 m

Hint:
- The relation between the number n of bounces and the hight h of the ball is an exponential function:
$h = f(n) = h_0 \cdot a^n$
where h = hight after n bounces, h_0 = initial hight = 12.8 m, a = decay factor = 0.75

9.16 $4'096

9.17 104'086

9.18 $5.10 \cdot 10^{20}$

9.19 $r = \sqrt[20]{2} - 1 = 3.5\%$ (rounded)

9.20 * a) $y = 10$ for $t = 0$

b) …

c) $y \to 2500$ as $t \to \infty$

9.21 * a) $y = f(x) = 2.58 \cdot 1.043^x$

b) $y(110) = 264.79$

9.22 a) 4th statement

b) 4th statement

c) 4th statement