

## Review exercises 2      Differential calculus, integral calculus

### Problems

R2.1 Decide whether the following statements are true or false:

- a) "The derivative of a function is a function."
- b) "The derivative of a function at a particular value of the variable is a number."
- c) "The function  $f$  has a relative maximum at  $x = x_1$  if  $f'(x_1) = 0$  and  $f''(x_1) > 0$ ."
- d) "If  $f''(x_2) = 0$  and  $f'''(x_2) < 0$ , then the function  $f$  has a point of inflection at  $x = x_2$ ."
- e) "Suppose that the function  $f$  has a relative maximum at  $x = x_1$ . If there are no other relative maxima and if there is no relative minimum at all, then the relative maximum is the absolute maximum of  $f$ ."
- f) "If  $g' = f$ , then  $g$  is an antiderivative of  $f$ ."
- g) "f with  $f(x) = 2x + 20$  is an antiderivative of  $g$  with  $g(x) = x^2$ ."
- h) "f with  $f(x) = 3x$  has infinitely many antiderivatives."
- i) "The indefinite integral of a function is a set of functions."

R2.2 Determine the value  $f(x_0)$ , the first derivative  $f'(x_0)$ , and the second derivative  $f''(x_0)$  at  $x_0$  for the following functions f:

- a)  $f(x) = 4x^2(x^2 - 1)$ 
  - i)  $x_0 = 0$
  - ii)  $x_0 = -1$
- b)  $f(x) = (-3x^2 + 2x - 1) \cdot e^x$ 
  - i)  $x_0 = 0$
  - ii)  $x_0 = -2$
- c)  $f(x) = (x^2 + 2) \cdot e^{-3x}$ 
  - i)  $x_0 = 1$
  - ii)  $x_0 = -\frac{1}{3}$

R2.3 For the given cost function  $C(x)$  and revenue function  $R(x)$  determine ...

- i) ... the marginal cost function  $C'(x)$ .
- ii) ... the marginal revenue function  $R'(x)$ .
- iii) ... the marginal profit function  $P'(x)$ .
- a)  $C(x) = 200 + 40x$                        $R(x) = 60x$
- b)  $C(x) = 100 + 20x + 5x^2$             $R(x) = 100x - 2x^2$
- c)  $C(x) = 50 + 20x^2 + 3e^{4x}$          $R(x) = 200x - e^{-4x^2}$

R2.4 For each function, find ...

- i) ... the relative maxima and minima.
- ii) ... the points of inflection.
- a)  $f(x) = 2x^3 - 9x^2 + 12x - 1$
- b)  $f(x)$  as in R2.2 a)

R2.5 The total revenue function for a commodity is given by

$$R(x) = 36x - 0.01x^2$$

Find the maximum revenue ...

- a) ... if production is not limited to a certain number of units.
- b) ... if production is limited to at most 1500 units.

R2.6 If the total cost function for a product is

$$C(x) = 100 + x^2$$

producing how many units  $x$  will result in a minimum average cost per unit? Find the minimum average cost.

R2.7 A firm can produce only 1000 units per month. The monthly total cost is given by

$$C(x) = 300 + 200x$$

dollars, where  $x$  is the number produced. If the total revenue is given by

$$R(x) = 250x - \frac{1}{100}x^2$$

dollars, how many items should the firm produce for maximum profit? Find the maximum profit.

R2.8 Determine the indefinite integrals below:

a)  $\int (x^4 - 3x^3 - 6) \, dx$

b)  $\int \left( \frac{1}{2}x^6 - \frac{2}{3x^4} \right) \, dx$

R2.9 The equation of the third derivative  $f'''$  of a function  $f$  is given as follows:

$$f'''(x) = 3x + 1$$

Find the equation of the function  $f$  such that  $f''(0) = 0$ ,  $f'(0) = 1$ ,  $f(0) = 2$

R2.10 If the marginal cost (in dollars) for producing a product is  $C'(x) = 5x + 10$ , with a fixed cost of \$800, what will be the cost of producing 20 units?

R2.11 A certain firm's marginal cost  $C'(x)$  and the derivative of the average revenue  $\bar{R}'(x)$  are given as follows:

$$C'(x) = 6x + 60$$

$$\bar{R}'(x) = -1$$

The total cost and revenue of the production of 10 items are \$1000 and \$1700, respectively.

How many units will result in a maximum profit? Find the maximum profit.

R2.12 The demand function for a product is

$$p = f(x) = 49 - x^2$$

and the supply function is

$$p = g(x) = 4x + 4$$

Find the equilibrium point and both the consumer's and the producer's surplus there.

R2.13 The demand function for a product is

$$p = f(x) = 110 - ax^2$$

and the supply function is

$$p = g(x) = 2 - \frac{6}{5}x + bx^2$$

with unknown parameters  $a$  and  $b$ . The equilibrium price is \$10, and the producer's surplus is \$73.33

Determine the two unknown parameters  $a$  and  $b$ .



b)  $f(x) = 4x^2(x^2 - 1)$   
 $f'(x) = 16x^3 - 8x = 8x(2x^2 - 1)$   
 $f''(x) = 48x^2 - 8 = 8(6x^2 - 1)$   
 $f'''(x) = 96x$

i)  $f'(x) = 0$  at  $x_1 = 0$ ,  $x_2 = \frac{1}{\sqrt{2}}$ , and  $x_3 = -\frac{1}{\sqrt{2}}$   
 $f''(x_1) = -8 < 0 \Rightarrow$  relative maximum at  $x_1 = 0$   
 $f''(x_2) = 16 > 0 \Rightarrow$  relative minimum at  $x_2 = \frac{1}{\sqrt{2}}$   
 $f''(x_3) = 16 > 0 \Rightarrow$  relative minimum at  $x_3 = -\frac{1}{\sqrt{2}}$

ii)  $f''(x) = 0$  at  $x_3 = \frac{1}{\sqrt{6}}$   
 $f'''(x_3) = \frac{96}{\sqrt{6}} \neq 0 \Rightarrow$  point of inflection at  $x_3 = \frac{1}{\sqrt{6}}$

- R2.5 a) **Relative** maximum at  $x_1 = 1800$   
 $R(x_1) = \$32'400$   
 $R(x) < R(x_1)$  if  $x \neq x_1$  as there is no relative minimum  
 $\Rightarrow R = \$32'400$  is the **absolute** maximum revenue at  $x = 1800$ .
- b) Relative maximum at  $x = 1800$  lies outside the possible interval  $0 \leq x \leq 1500$   
 $R(1500) = \$31'500 > R(0) = 0\$$   
 $\Rightarrow R = \$31'500$  is the **absolute** maximum revenue at  $x = 1500$ .

R2.6  $\bar{C}(x) = \frac{C(x)}{x} = \frac{100}{x} + x$   
 $\bar{C}(x)$  has a **relative** minimum at  $x_1 = 10$   
 $\bar{C}(10) = \$20$   
 $\bar{C}(x) > \bar{C}(x_1)$  if  $x \neq x_1$  as there is no relative maximum  
 $\Rightarrow \bar{C} = \$20$  is the **absolute** minimum average cost at  $x = 10$ .

R2.7  $P(x) = R(x) - C(x) = -\frac{1}{100}x^2 + 50x - 300$   
 $P(x)$  has a **relative** maximum at  $x_1 = 2500$ . This is outside the possible interval  $0 \leq x \leq 1000$   
 $P(1000) = \$39'700 > P(0) = -300\$$   
 $\Rightarrow P = \$39'700$  is the **absolute** maximum profit at the endpoint  $x = 1000$ .

R2.8 a)  $\int (x^4 - 3x^3 - 6) dx = \frac{x^5}{5} + \frac{3x^4}{4} - 6x + C$

b)  $\int \left( \frac{1}{2}x^6 - \frac{2}{3x^4} \right) dx = \frac{x^7}{14} + \frac{2}{9x^3} + C$

R2.9  $f(x) = \frac{x^4}{8} + \frac{x^3}{6} + x + 2$

R2.10  $C(20) = \$2000$

Hint:

- First, determine the cost function  $C(x) \Rightarrow C(x) = \frac{5}{2}x^2 + 10x + 800$

R2.11  $P = \$800$  is the absolute maximum profit at  $x = 15$  units.

Hints:

- Determine the cost function  $C(x) \Rightarrow C(x) = 3x^2 + 60x + 100$

- Determine the average revenue function  $\bar{R}(x) \Rightarrow \bar{R}(x) = -x + C$
- Determine the revenue function  $R(x) \Rightarrow R(x) = -x^2 + 180x$
- Find the profit function  $P(x) \Rightarrow P(x) = -4x^2 + 120x - 100$
- Find the relative maximum of the profit function  $P(x)$ .
- Check if the relative maximum is the absolute maximum.

R2.12    Equilibrium quantity     $x = 5$   
          Equilibrium price         $p = 24$   
          Consumer's surplus        $CS = 83.33$   
          Producer's surplus        $PS = 50$

R2.13     $a = 1$   
           $b = 0.2$