

Exercises 9 Exponential function and equations Compound interest, exponential function

Objectives

- be able to calculate the future capital that is invested at an interest rate which is compounded annually.
- be able to treat compound interest tasks.
- be able to graph an exponential function out of its equation.
- be able to determine the equation of an exponential function out of the coordinates of two points of the graph.
- be able to treat applied tasks by means of an exponential function.

Problems

- 9.1 Compound interest at an annual rate r is paid on an initial capital C_0 .
- Assume an initial capital $C_0 = 1000.00$ CHF, and an annual interest rate $r = 2\%$. Determine the capital after one, two, three, four, and five years' time.
 - Try to develop a formula which allows you to calculate the capital C_n after n years' time for any values of C_0 , r , and n .
- 9.2 What is the future capital if 8000 CHF is invested for 10 years at 12% compounded annually?
- 9.3 What present value amounts to 10'000 CHF if it is invested for 10 years at 6% compounded annually?
- 9.4 At what interest rate, compounded annually, would 10'000 CHF have to be invested to amount to 14'071 CHF in 7 years?
- 9.5 Ms Smith wants to invest 150'000 CHF for five years. Bank A offers an interest rate of 6.5% compounded annually. Bank B offers to pay 200'000 CHF after five years. Which bank makes the better offer?
- 9.6 The purchase of Alaska cost the United States \$ 7 million in 1869. If this money had been placed in a savings account paying 6% compounded annually, how much money would be available from this investment in 2020?
- 9.7 Mary Stahley invested \$2500 in a 36-month certificate of deposit (CD) that earned 8.5% annual simple interest. When the CD matured, she invested the full amount in a mutual fund that had an annual growth equivalent to 18% compounded annually. How much was the mutual fund worth after 9 years?
- 9.8 A capital is invested for 4 years at 4% and for 3 more years at 6%, compounded annually. Eventually, the capital amounts to 72'000 CHF.
- Determine the initial capital.
 - What is the average interest rate with respect to the whole period of time?
- 9.9 An unknown initial capital is invested at an unknown interest rate, compounded annually. After 2 years, the capital amounts to 5'891.74 CHF, and after another 5 years the capital is 6'997.54 CHF. Determine both initial capital and interest rate.

9.10 Look at the following exponential function:

$$\begin{aligned} f: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f(x) = 2^x \end{aligned}$$

- Establish a table of values of f for the interval $-3 \leq x \leq 3$.
- Draw the graph of f in the interval $-3 \leq x \leq 3$ into a Cartesian coordinate system.

9.11 Graph the following exponential functions into one coordinate system:

$$\begin{aligned} f_1: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f_1(x) = 2^x \end{aligned}$$

$$\begin{aligned} f_2: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f_2(x) = 0.2^x \end{aligned}$$

$$\begin{aligned} f_3: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f_3(x) = 3 \cdot 0.5^x \end{aligned}$$

$$\begin{aligned} f_4: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f_4(x) = -2 \cdot 3^x \end{aligned}$$

9.12 The graph of an exponential function contains the points P and Q. Determine the equation of the exponential function.

- $P(0|1.02)$ $Q(1|1.0302)$
- $P(1|12)$ $Q(3|192)$
- $P(0|10'000)$ $Q(5|777.6)$
- $P(5|16)$ $Q\left(9|\frac{1}{16}\right)$

9.13 A house that 20 years ago was worth \$160'000 has increased in value by 4% each year because of inflation. What is its worth today?

9.14 Suppose a country has a population of 20 million and projects a growth rate of 2% per year for the next 20 years. What will the population of this country be in 10 years?

9.15 A ball is dropped from a height of 12.8 meters. It rebounds $\frac{3}{4}$ of the height from which it falls every time it hits the ground. How high will the ball bounce after it strikes the ground for the fourth time?

9.16 A machine is valued at \$10'000. The depreciation at the end of each year is 20% of its value at the beginning of the year. Find its value at the end of 4 years.

9.17 The size of a certain bacteria culture grows exponentially. At 8 a.m. and 11 a.m. the number of bacteria was 2'300 and 18'400, respectively. Determine the number of bacteria at 1.30 p.m.

9.18 In a physical experiment the number of radioactive nuclei in a certain preparation decreases exponentially. 5 hours after the start of the experiment $1.56 \cdot 10^{16}$ nuclei were counted. 3 hours later, the number has fallen to $3.05 \cdot 10^{13}$. What was the number of nuclei at the beginning of the experiment?

9.19 A capital pays interest, compounded annually. What is the interest rate such that the capital doubles in 20 years?

9.20 * Suppose that the number y of otters t years after they were reintroduced into a wild and scenic river is given by

$$y = 2500 - 2490 \cdot e^{-0.1 \cdot t}$$

- Find the population when the otters were introduced.
- Draw the graph of the function $f: t \rightarrow y = f(t)$.
- What is the expected upper limit of the number of otters?

9.21 * The consumer price index (CPI) is calculated by averaging the prices of various items after assigning a weight to each item. The following table gives the consumer price indexes for selected years from 1940 through 2002:

Year	CPI	Year	CPI
1940	14.0	1980	82.4
1950	24.1	1990	130.7
1960	29.6	2000	172.2
1970	38.8	2002	179.9

- Find an equation that models these data, i.e. try to find the parameters a and c of the exponential function $f: x \mapsto y = f(x) = c \cdot a^x$ (x = years after 1900, y = CPI) that fits the data.
- Use the model to predict the CPI in 2010.

9.22 Decide which statements are true or false. Put a mark into the corresponding box. In each problem a) to c), exactly one statement is true.

- In a compound interest scheme ...
 - ... the graph that represents the growth of the capital is a parabola.
 - ... the interest paid at the end of each period only depends on the interest rate.
 - ... the interest rate depends on the capital of the previous period.
 - ... the capital grows exponentially.
- The graph of an exponential function ...
 - ... is a parabola.
 - ... is a straight line if the initial value is equal to zero.
 - ... never intersects the y-axis.
 - ... never touches the x-axis.
- If a quantity grows exponentially in time ...
 - ... the growth factor itself grows.
 - ... the growth factor depends on the initial value.
 - ... the quantity doubles in one year if the annual growth factor is 100%.
 - ... the quantity doubles in constant time intervals.

Answers

9.1 a) $C_0 = 1000.00 \text{ CHF}$ $C_1 = 1020.00 \text{ CHF}$ $C_2 = 1040.40 \text{ CHF}$
 $C_3 = 1061.21 \text{ CHF}$ $C_4 = 1082.43 \text{ CHF}$ $C_5 = 1104.08 \text{ CHF}$

b) $C_n = C_0 (1 + r)^n$

9.2 $C_{10} = 24'846.79 \text{ CHF}$

9.3 $C_0 = 5'583.95 \text{ CHF}$

9.4 $r = 5\%$

9.5 Bank A: $C_5 = 205'513.00 \text{ CHF}$
Bank B: $C_5 = 200'000.00 \text{ CHF}$

9.6 $C_{151} = \$ 46'375 \text{ million (rounded to millions)}$

9.7 $\$13'916.24$

2 periods: 3 years of simple interest, 9 years of compound interest

- 3 years of simple interest:

$C_n = C_0(1 + nr)$ where $C_0 = \$2500$, $n = 3$, $r = 8.5\% = 0.085$
 $\Rightarrow C_3 = \$3137.50$

- 9 years of compound interest:

$C_n = C_0 q^n$ where $C_0 = \dots (= C_3 \text{ after first 3 years})$, $q = 1 + 18\% = 1.18$, $n = 9$
 $\Rightarrow C_9 = \$13'916.24$

9.8 a) $C_0 = 51'675 \text{ CHF}$

Hints:

- First, look at the second period (3 years, starting after 4 years from now), and calculate the capital at the beginning of this second period.
- Then, calculate the initial capital.

b) $r = 4.85\%$

Hint:

- The average interest rate r must be such that

$C_n = C_0 q^n$ where $C_0 = \text{initial capital}$, $C_n = \text{capital after the whole 7 years}$, $n = 7$, $q = 1 + r$

9.9 $r = 3.5\%$, $C_0 = 5'500.00 \text{ CHF}$

Hints:

- First, look at the second period of 5 years, where $C_0 = 5'891.74 \text{ CHF}$ and $C_5 = 6'997.54 \text{ CHF}$
- The $5'891.74 \text{ CHF}$ are the C_2 at the end of the first 2 years.

9.10 ...

9.11 ...

9.12 a) $y = f(x) = 1.02 \cdot 1.01^x$

Hints:

- The equation of an exponential function is $y = f(x) = c \cdot a^x$

- If $P(0|1.02)$ and $Q(1|1.0302)$ are points of the graph of the exponential function, their coordinates must fulfil the equation of the exponential function, i.e. $1.02 = f(0) = c \cdot a^0$ and $1.0302 = f(1) = c \cdot a^1$

- Solve the two equations for c and a .

b) $y = f(x) = 3 \cdot 4^x$

c) $y = f(x) = 10'000 \cdot 0.6^x$

d) $y = f(x) = 16'384 \cdot 0.25^x$

9.13 \$350'580 (rounded)

Hint:

- The relation between time t and the value V of the house is an exponential function:

$$V = f(t) = V_0 \cdot a^t$$

where V = value after time t , V_0 = initial value (at $t = 0$) = \$160'000, a = growth factor = $1 + 4\% = 1.04$

9.14 24.4 million (rounded)

9.15 4.05 m

Hint:

- The relation between the number n of bounces and the height h of the ball is an exponential function:

$$h = f(n) = h_0 \cdot a^n$$

where h = height after n bounces, h_0 = initial height = 12.8 m, a = decay factor = 0.75

9.16 \$4'096

9.17 104'086

9.18 $5.10 \cdot 10^{20}$

9.19 $r = \sqrt[20]{2} - 1 = 3.5\%$ (rounded)

9.20 * a) $y = 10$ for $t = 0$

b) ...

c) $y \rightarrow 2500$ as $t \rightarrow \infty$

9.21 * a) $y = f(x) = 2.58 \cdot 1.043^x$

b) $y(110) = 264.79$

9.22 a) 4th statement

b) 4th statement

c) 4th statement