Exercises 15 Applications of differential calculus
Relative maxima/minima, points of inflection

Objectives

- be able to determine the relative maxima/minima of a function.
- be able to determine the points of inflection of a function.
- be able to find the absolute maximum/minimum of a cost/revenue/profit function.
- be able to find the absolute minimum of an average cost function.
- be able to find the point of diminishing returns of a profit function.

Problems

15.1 For each function, find …

i) … the relative maxima and minima.

ii) … the points of inflection.

a) $f(x) = x^2 - 4$

b) $f(x) = - 8x^3 + 12x^2 + 18x$

c) $s(t) = t^4 - 8t^2 + 16$

d) $f(x) = x\, e^{-x}$

e) * $f(x) = (1 - e^{-2x})^2$

f) * $V(r) = - D\left(\frac{2a}{r} - \frac{a^2}{r^2}\right)$ $(D > 0,\ a > 0)$

15.2 The total revenue (in dollars) for a firm is given by

$$R(x) = 8000x - 40x^2 - x^3$$

where x is the number of units sold per day. If only 50 units can be sold per day, find the number of units that must be sold to maximise revenue. Find the maximum revenue.

Hints:
- First, find the relative maximum.
- Then, check if the relative maximum is the absolute maximum.

15.3 If the total revenue (in dollars) for a commodity is

$$R(x) = 2000x + 20x^2 - x^3$$

where x is the number of items sold, …

a) … find the level of sales, x, that maximises revenue, and find the maximum revenue.

b) … find the maximum average revenue.

15.4 If the total cost (in dollars) for a commodity is given by

$$C(x) = \frac{1}{4}x^2 + 4x + 100$$

where x represents the number of units produced, producing how many units will result in a minimum average cost per unit? Find the minimum average cost.

15.5 Suppose that the production capacity for a certain commodity cannot exceed 30. If the total profit (in dollars) for this company is

$$P(x) = 4x^3 - 210x^2 + 3600x - 200$$

where x is the number of units sold, find the number of items that will maximize profit.

15.6 Suppose the annual profit for a store (in thousands of dollars) is given by

$$P(x) = - 0.1x^3 + 3x^2 + 6$$

where x is the number of years past 2000. If this model is accurate, find the point of diminishing returns for the profit.

15.7 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) If f has a relative maximum at $x = x_0$ it can be concluded that ...

- ☐ ... $f(x_0) > f(x)$ for any $x \neq x_0$
- ☐ ... $f(x_0) > f(x)$ for any $x > x_0$
- ☐ ... $f(x_0) > f(x)$ for any $x < x_0$
- ☐ ... $f(x_0) > f(x)$ for all x which are in a certain neighbourhood of x_0

b) If $f(x_0) < 0$, $f'(x_0) = 0$, and $f''(x_0) \neq 0$, it can be concluded that f has ...

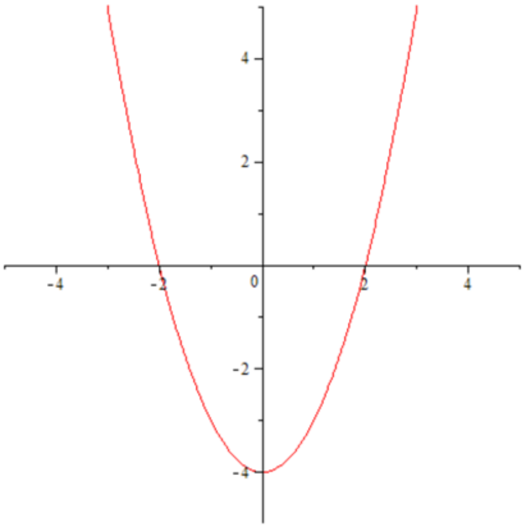
- ☐ ... no relative minimum at $x = x_0$
- ☐ ... no relative maximum at $x = x_0$
- ☐ ... no point of inflection at $x = x_0$
- ☐ ... a point of inflection at $x = x_0$

c) The absolute maximum of a function ...

- ☐ ... is always a relative maximum.
- ☐ ... can be a relative minimum.
- ☐ ... can be a relative maximum.
- ☐ ... always exists.

Answers

15.1 a) $f(x) = x^2 - 4$

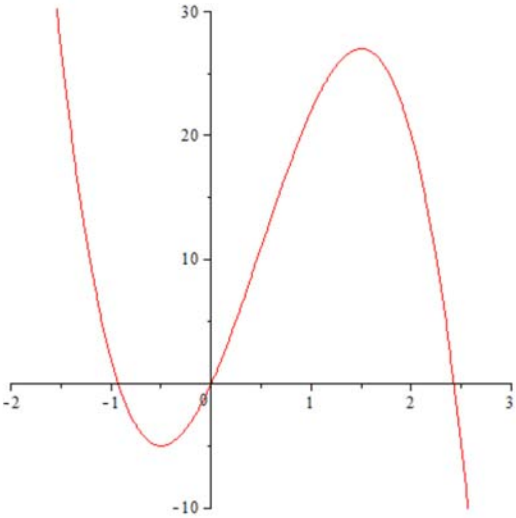


$f'(x) = 2x$
$f''(x) = 2$
$f'''(x) = 0$

i) $f'(x) = 0$ at $x_1 = 0$
$f''(x_1) = 2 > 0$ ⇒ relative minimum at $x_1 = 0$
no relative maximum

ii) $f''(x) = 2 \neq 0$ for all x ⇒ no point of inflection

b) $f(x) = -8x^3 + 12x^2 + 18x$



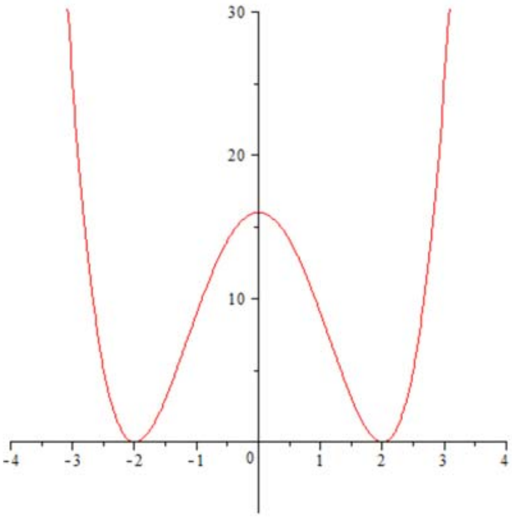
$f'(x) = -24x^2 + 24x + 18$
$f''(x) = -48x + 24$
$f'''(x) = -48$

i) $f'(x) = 0$ at $x_1 = -\frac{1}{2}$ and $x_2 = \frac{3}{2}$
$f''(x_1) = 48 > 0$ ⇒ relative minimum at $x_1 = -\frac{1}{2}$
$f''(x_2) = -48 < 0$ ⇒ relative maximum at $x_2 = \frac{3}{2}$

ii) $f''(x) = 0$ at $x_3 = \frac{1}{2}$

$f'''(x_3) = -48 \neq 0$ ⇒ point of inflection at $x_3 = \frac{1}{2}$

c) $s(t) = t^4 - 8t^2 + 16$



$s'(t) = 4t^3 - 16t$
$s''(t) = 12t^2 - 16$
$s'''(t) = 24t$

i) $s'(t) = 0$ at $t_1 = 0$, $t_2 = -2$, and $t_3 = 2$

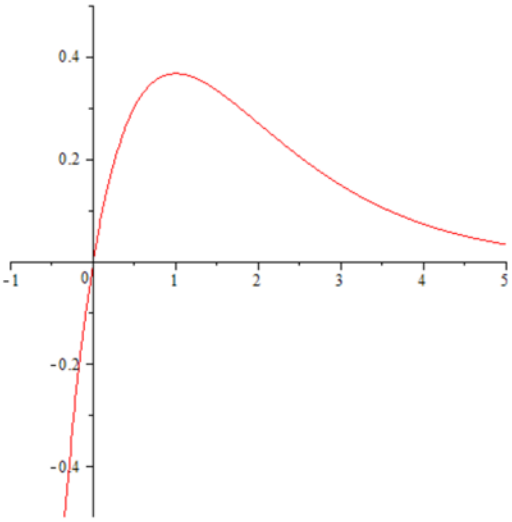
$s''(t_1) = -16 < 0$ ⇒ relative maximum at $t_1 = 0$
$s''(t_2) = 32 > 0$ ⇒ relative minimum at $t_2 = -2$
$s''(t_3) = 32 > 0$ ⇒ relative minimum at $t_3 = 2$

ii) $s''(t) = 0$ at $t_4 = -\frac{2}{\sqrt{3}}$ and $t_5 = \frac{2}{\sqrt{3}}$

$s'''(t_4) = -\frac{48}{\sqrt{3}} \neq 0$ ⇒ point of inflection at $t_4 = -\frac{2}{\sqrt{3}}$

$s'''(t_5) = \frac{48}{\sqrt{3}} \neq 0$ ⇒ point of inflection at $t_5 = \frac{2}{\sqrt{3}}$

d) $f(x) = x\,e^{-x}$

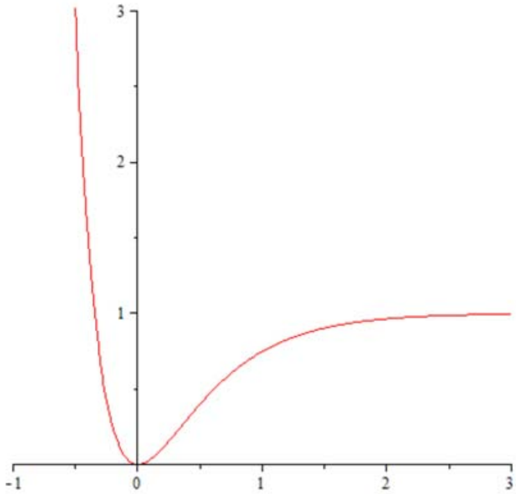


$f'(x) = e^{-x} - x\,e^{-x} = (1 - x)\,e^{-x}$
$f''(x) = -e^{-x} - (1 - x)\,e^{-x} = (x - 2)\,e^{-x}$
$f'''(x) = e^{-x} - (x - 2)\,e^{-x} = (3 - x)\,e^{-x}$

i) $f'(x) = 0$ at $x_1 = 1$
$f''(x_1) = -\frac{1}{e} < 0$ ⇒ relative maximum at $x_1 = 1$
no relative minimum

ii) $f''(x) = 0$ at $x_2 = 2$
$f'''(x_2) = \frac{1}{e^2} \neq 0$ ⇒ point of inflection at $x_2 = 2$

e) * $f(x) = (1 - e^{-2x})^2$



$f'(x) = 2(1 - e^{-2x}) \cdot 2e^{-2x} = 4(1 - e^{-2x})\, e^{-2x}$
$f''(x) = 4\left(2e^{-2x}e^{-2x} + (1 - e^{-2x})(- 2e^{-2x})\right) = 8e^{-2x}(2e^{-2x} - 1)$
$f'''(x) = 8\left(- 2e^{-2x}(2e^{-2x} - 1) + e^{-2x}(- 4e^{-2x})\right) = 16e^{-2x}(1 - 4e^{-2x})$

i) $f'(x) = 0$ at $x_1 = 0$
$f''(x_1) = 8 > 0$ ⇒ relative minimum at $x_1 = 0$
no relative maximum

ii) $f''(x) = 0$ at $x_2 = \frac{\ln(2)}{2} = 0.34\ldots$
$f'''(x_2) = - 8 \neq 0$ ⇒ point of inflection at $x_2 = 0.34\ldots$

f) * $V'(r) = - D\left(-\frac{2a}{r^2} + \frac{2a^2}{r^3}\right) = \frac{2aD}{r^2}\left(1 - \frac{a}{r}\right)$
$V''(r) = - D\left(\frac{4a}{r^3} - \frac{6a^2}{r^4}\right) = \frac{2aD}{r^3}\left(\frac{3a}{r} - 2\right)$
$V'''(r) = - D\left(-\frac{12a}{r^4} + \frac{24a^2}{r^5}\right) = \frac{12aD}{r^4}\left(1 - \frac{2a}{r}\right)$

i) $V'(r) = 0$ at $r_1 = a$
$V''(r_1) = \frac{2D}{a^2} > 0$ ⇒ relative minimum at $r_1 = a$
no relative maximum

ii) $V''(r) = 0$ at $r_2 = \frac{3a}{2}$
$V'''(r_2) = -\frac{64D}{81a^3} \neq 0$ ⇒ point of inflection at $r_2 = \frac{3a}{2}$

15.2 **Relative** maximum at $x_1 = 40$
$R(x_1) = \$192'000$
$R(x) < R(x_1)$ if $x \neq x_1$ as there is no relative minimum
⇒ R = \$192'000 is the **absolute** maximum revenue at x = 40.

15.3 a) **Relative** maximum at $x_1 = \frac{100}{3}$ → 33 or 34
R(33) = \$51'843

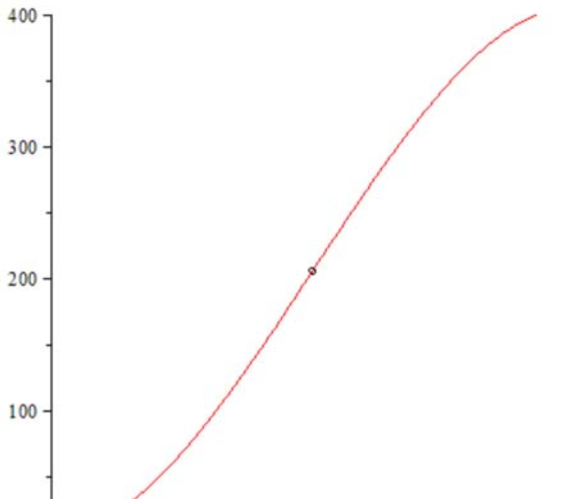
$R(34) = \$51'816$
$R(x) < R(x_1)$ if $x \neq x_1$ as there is no relative minimum
$\Rightarrow$ $R = \$51'843$ is the **absolute** maximum revenue at $x = 33$.

b) $\overline{R}(x) = \frac{R(x)}{x} = 2000 + 20x - x^2$
$\overline{R}(x)$ has a **relative** maximum at $x_2 = 10$
$\overline{R}(10) = \$2100$
$\overline{R}(x) < \overline{R}(x_2)$ if $x \neq x_2$ as there is no relative minimum
$\Rightarrow$ $\overline{R} = \$2100$ is the **absolute** maximum average revenue at $x = 10$.

15.4 $\overline{C}(x) = \frac{C(x)}{x} = \frac{1}{4}x + 4 + \frac{100}{x}$
$\overline{C}(x)$ has a **relative** minimum at $x_1 = 20$
$\overline{C}(20) = \$14$
$\overline{C}(x) > \overline{C}(x_1)$ if $x \neq x_1$ as there is no relative maximum
$\Rightarrow$ $\overline{C} = \$14$ is the **absolute** minimum average cost at $x = 20$.

15.5 $P(x)$ has a **relative** maximum at $x_1 = 15$ and a **relative** minimum at $x_2 = 20$.
$P(x_1) = \$20'050$
$P(x) < P(x_1)$ if $x < x_1$ as there is no relative minimum on the interval $x < x_1$
$P(30) = \$26'800 > \$20'050$ (!)
$\Rightarrow$ $P = \$26'800$ is the **absolute** maximum profit at the endpoint $x = 30$.

15.6 $P(x)$ has a point of inflection at $x_1 = 10$



$P(10) = 206$
$\Rightarrow$ point of diminishing returns (10|206), i.e. when $x = 10$ (in the year 2010) and $P = \$206'000$.

15.7 a) 4th statement

b) 3rd statement

c) 3rd statement