

Exercises 14 Differentiation rules Coefficient/sum/product rule, chain rule, higher-order derivatives

Objectives

- be able to apply the coefficient, sum, product rule to determine the derivative of a function.
- be able to apply the chain rule to determine the derivative of a function.
- be able to determine a higher-order derivative of a function.

Problems

14.1 Determine the derivative by applying the **coefficient rule**:

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|-------------------------------------|------------------------------|-------------------------|
| a) $f(x) = 3x^5$ | b) $f(x) = -4x^3$ | c) $f(x) = -x^{10}$ |
| d) $f(x) = a \cdot x^3$ | e) $f(x) = n \cdot x^{n-1}$ | f) $f(x) = 9 \cdot 3^x$ |
| g) $s(t) = \frac{1}{2} g \cdot t^2$ | h) $S(T) = \alpha \cdot T^4$ | i) $C(x) = (-3x)^3$ |

14.2 Determine the derivative by applying the **sum rule**:

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|---|--|---|
| a) $f(x) = x^5 + x^6$ | b) $f(x) = x^{10} - x^9$ | c) $f(x) = 1 + x + 3x^3$ |
| d) $f(x) = \frac{1}{4}x^4 + 3x^2 - 2$ | e) $f(x) = 3x^2(x - 2)$ | f) $f(x) = -3x^8 + x^5 - 3x + 99$ |
| g) $f(x) = ax^2 + bx + c$ | h) $f(x) = 3(a^2 - 2ax + x^2)$ | i) $f(x) = \frac{x^3}{3} - \frac{3}{x^3}$ |
| j) $s(t) = s_0 + v_0 t + \frac{1}{2} g \cdot t^2$ | k) $V(r) = -\frac{a}{r} + \frac{b}{r^2}$ | l) $C(n) = C_0(1 + nr)$ |

14.3 Determine the derivative by applying the **product rule**:

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|--|--|
| a) $f(x) = x \cdot e^x$ | b) $f(x) = x^3 \cdot 3^x$ |
| c) $f(x) = -2x^5(x - 1)$ | d) $f(x) = (2x - 1) \cdot e^x$ |
| e) $f(x) = (2x - 1)(-3x^2 - x + 1)$ | f) $f(x) = 3(1 - x^2)(x^{10} - x^9)$ |
| g) $V(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3} \right)$ | h) $T(V) = \frac{1}{n \cdot R} \left(p + \frac{a \cdot n^2}{V^2} \right) (V - n \cdot b)$ |

14.4 Determine the derivative by applying the **chain rule**:

- | | | |
|----------------------|------------------------------|----------------------------------|
| a) $f(x) = (2x)^3$ | b) $f(x) = (3x - 1)^5$ | c) $f(x) = (-3x^3 + x^2 - 4x)^6$ |
| d) $f(x) = e^{4x}$ | e) $f(x) = e^{-x}$ | f) $f(x) = e^{1 - \frac{x}{2}}$ |
| g) $f(x) = e^{-x^2}$ | h) $f(x) = e^{x^2 - 2x + 5}$ | i) $f(x) = e^{e^x}$ |
| j) $f(x) = 2^{3^x}$ | k) * $f(x) = 2^{e^{2x}}$ | l) ** $f(x) = x^x$ |

14.5 Determine the derivative by applying the appropriate differentiation rule(s), and simplify the expression as far as possible:

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|---|-------------------------------------|
| a) $f(x) = (x - 2) e^{2x}$ | b) $f(x) = (2 - x^2) e^{-x}$ |
| c) $f(x) = (3x^3 - 2x^2 + x - 1) e^{-2x}$ | d) $f(x) = (x - 2)^2 e^{-x^2 - 2x}$ |
| e) $f(x) = ax e^{-\frac{x^2}{2}}$ | f) $P(v) = av^2 e^{-bv^2}$ |

14.6 Determine the derivative of the indicated function at the indicated value of the variable:

- | | | | | | |
|----|--------------|----------|----|--------------|---------|
| a) | f in 14.1 b) | $x = 2$ | b) | s in 14.1 g) | $t = 4$ |
| c) | f in 14.2 g) | $x = -1$ | d) | f in 14.5 e) | $x = 0$ |

14.7 Determine the second and third derivatives of the functions in problem ...

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|----|-------------|----|-------------|
| a) | ... 14.1 a) | b) | ... 14.2 g) |
| c) | ... 14.3 a) | d) | ... 14.4 g) |
| e) | ... 14.5 b) | f) | ... 14.5 e) |

14.8 Determine the indicated higher-order derivatives:

- a) $f''(-1)$ with function f in 14.1 a)
Hint:
- You have already determined $f'(x)$ in 14.7 a).
- b) $f'''(2)$ with function f in 14.5 e)
Hint:
- You have already determined $f''(x)$ in 14.7 f).

Answers

- 14.1 a) $f'(x) = 3 \cdot 5x^4 = 15x^4$
 b) $f'(x) = (-4) 3x^2 = -12x^2$
 c) $f'(x) = (-1) 10x^9 = -10x^9$
 d) $f'(x) = a \cdot 3x^2 = 3ax^2$

Hint:

- a is a constant.

- e) $f'(x) = n(n-1)x^{n-2}$
 f) $f'(x) = 9 \cdot 3^x \cdot \ln(3)$
 g) $s'(t) = \frac{g}{2} 2t = gt$

Hints:

- The name of the function is s, and the variable is t.
 - g is a constant.

- h) $S'(T) = \alpha \cdot 4T^3 = 4\alpha T^3$
 i) $C'(x) = -81x^2$

- 14.2 a) $f'(x) = 5x^4 + 6x^5$ b) $f'(x) = 10x^9 - 9x^8$ c) $f'(x) = 1 + 9x^2$
 d) $f'(x) = x^3 + 6x$ e) $f'(x) = 9x^2 - 12x$ f) $f'(x) = -24x^7 + 5x^4 - 3$
 g) $f'(x) = 2ax + b$ h) $f'(x) = -6a + 6x$ i) $f'(x) = x^2 + \frac{9}{x^4}$
 j) $s'(t) = v_0 + gt$ k) $V'(r) = \frac{a}{r^2} - \frac{2b}{r^3}$ l) $C'(n) = C_0 r$

- 14.3 a) $f'(x) = e^x + x \cdot e^x$
 b) $f'(x) = 3x^2 \cdot 3^x + x^3 \cdot 3^x \cdot \ln(3)$
 c) $f'(x) = -2(5x^4(x-1) + x^5)$
 d) $f'(x) = 2 \cdot e^x + (2x-1) \cdot e^x$
 e) $f'(x) = 2(-3x^2 - x + 1) + (2x-1)(-6x-1)$
 f) $f'(x) = 3(-2x(x^{10} - x^9) + (1-x^2)(10x^9 - 9x^8))$
 g) $V'(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3} \right) + e^r \left(2a \cdot r + \frac{3b}{r^4} \right)$

Hints:

- V is the name of the function, and r is the variable.
 - a and b are constants.

- h) $T'(V) = \frac{1}{n \cdot R} \left(-\frac{2a \cdot n^2}{V^3} (V - n \cdot b) + \left(p + \frac{a \cdot n^2}{V^2} \right) \right)$

Hints:

- T is the name of the function, and V is the variable.
 - n, R, p, a and b are constants.

- 14.4 a) $f(x) = 3(2x)^2 \cdot 2 = 24x^2$ b) $f(x) = 5(3x-1)^4 \cdot 3 = 15(3x-1)^4$
 c) $f(x) = 6(-3x^3 + x^2 - 4x)^5 \cdot (-9x^2 + 2x - 4)$ d) $f(x) = e^{4x} 4 = 4e^{4x}$
 e) $f(x) = e^{-x} (-1) = -e^{-x}$ f) $f(x) = e^{1-\frac{x}{2}} \left(-\frac{1}{2} \right) = -\frac{1}{2} e^{1-\frac{x}{2}}$

- g) $f(x) = e^{-x^2}(-2x) = -2x \cdot e^{-x^2}$ h) $f(x) = e^{x^2-2x+5}(2x-2)$
 i) $f(x) = e^{e^x} \cdot e^x$ j) $f(x) = 2^{3^x} \cdot \ln(2) \cdot 3^x \cdot \ln(3)$
 k) * $f(x) = 2^{e^{2x}} \cdot \ln(2) \cdot e^{2x} \cdot 2$
 l) ** $f(x) = x^x \cdot (\ln(x) + 1)$

Hints:

- The expression x^x can be rewritten as follows: $x^x = e^{\ln(x^x)} = e^{x \cdot \ln(x)}$
- The derivative of $\ln(x)$ is $\frac{1}{x}$

- 14.5 a) $f'(x) = e^{2x} + (x-2)e^{2x} = (2x-3)e^{2x}$
 b) $f'(x) = -2xe^{-x} + (2-x^2)e^{-x}(-1) = (x^2-2x-2)e^{-x}$
 c) $f'(x) = (9x^2-4x+1)e^{-2x} - 2(3x^3-2x^2+x-1)e^{-2x} = (-6x^3+13x^2-6x+3)e^{-2x}$
 d) $f'(x) = 2(x-2)e^{-x^2-2x} + (x-2)^2(-2x-2)e^{-x^2-2x} = -2(x^3-3x^2-x+6)e^{-x^2-2x}$
 e) $f'(x) = a \left(e^{-\frac{x^2}{2}} + x e^{-\frac{x^2}{2}}(-x) \right) = a(1-x^2)e^{-\frac{x^2}{2}}$
 f) $P'(v) = a(2v e^{-bv^2} + v^2 e^{-bv^2}(-2bv)) = 2av(1-bv^2)e^{-bv^2}$

- 14.6 a) $f'(2) = -48$ b) $s'(4) = 4g$
 c) $f'(-1) = -2a + b$ d) $f'(0) = a$

- 14.7 a) 14.1 a)
 $f''(x) = 15 \cdot 4x^3 = 60x^3$
 $f'''(x) = 60 \cdot 3x^2 = 180x^2$
 b) 14.2 g)
 $f''(x) = 2a \cdot 1 = 2a$
 $f'''(x) = 0$
 c) 14.3 a)
 $f''(x) = e^x + (e^x + x \cdot e^x) = (x+2)e^x$
 $f'''(x) = e^x + (x+2)e^x = (x+3)e^x$
 d) 14.4 g)
 $f''(x) = -2(e^{-x^2} + x e^{-x^2}(-2x)) = 2(2x^2-1)e^{-x^2}$
 $f'''(x) = 2(4x e^{-x^2} + (2x^2-1)e^{-x^2}(-2x)) = 4x(-2x^2+3)e^{-x^2}$
 e) 14.5 b)
 $f''(x) = (2x-2)e^{-x} + (x^2-2x-2)e^{-x}(-1) = (4x-x^2)e^{-x}$
 $f'''(x) = (4-2x)e^{-x} + (4x-x^2)e^{-x}(-1) = (x^2-6x+4)e^{-x}$
 f) 14.5 e)
 $f''(x) = a \left(-2x e^{-\frac{x^2}{2}} + (1-x^2)e^{-\frac{x^2}{2}}(-x) \right) = a(x^3-3x)e^{-\frac{x^2}{2}}$
 $f'''(x) = a \left((3x^2-3)e^{-\frac{x^2}{2}} + (x^3-3x)e^{-\frac{x^2}{2}}(-x) \right) = a(-x^4+6x^2-3)e^{-\frac{x^2}{2}}$

- 14.8 a) $f''(-1) = -60$
 b) $f'''(2) = a(-16+6 \cdot 4-3)e^{-\frac{4}{2}} = \frac{5a}{e^2}$