Exercises 13 — Derivative
Rate of change, derivative of constant/power/exponential functions

Objectives

- be able to state the rate of change of a constant/linear function.
- be able to determine a rate of change of a basic power/exponential function.
- be able to estimate a rate of change out of the graph of a function.
- be able to determine the derivative of a constant/linear function.
- be able to determine the derivative of a basic power/exponential function.

Problems

13.1 For each of the following functions $f: \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow y = f(x) = ...$

i) ... draw the graph of f.

ii) ... state the rate of change $f'(x_0)$ at the given x_0.

a)	$f(x) = 3$	$x_0 = 2$
b)	$f(x) = c \ (c \in \mathbb{R})$	any $x_0 \in \mathbb{R}$
c)	$f(x) = 2x - 3$	$x_0 = 4$
d)	$f(x) = mx + q \ (m \in \mathbb{R} \setminus \{0\}, q \in \mathbb{R})$	any $x_0 \in \mathbb{R}$
e) *	$f(x) = \lvert x \rvert$	any $x_0 \in \mathbb{R}$

13.2 For each of the functions in problem 13.1. ...

i) ... determine the derivative f' (domain, codomain, equation).

ii) ... draw the graph of f'.

13.3 Look at the function f and its derivative f':

$$f: D \rightarrow \mathbb{R}$$
$$x \rightarrow y = f(x) = 24\sqrt{x+1} - 2x - 60$$

$$f': D_1 \rightarrow \mathbb{R}$$
$$x \rightarrow y = f'(x) = \frac{12}{\sqrt{x+1}} - 2$$

Determine the largest possible …

a) … domain D of f.

b) … domain D_1 of f'.

13.4 Determine f'(x):

a)	$f(x) = 3$	b)	$f(x) = 0$	c)	$f(x) = -1$
d)	$f(x) = x^3$	e)	$f(x) = x^4$	f)	$f(x) = x^5$
g)	$f(x) = x^{17}$	h)	$f(x) = x^{200}$	i)	$f(x) = x^{100'001}$
j)	$f(x) = x^{-1}$	k)	$f(x) = x^{-2}$	l)	$f(x) = x^{-17}$
m)	$f(x) = \frac{1}{x}$	n)	$f(x) = \frac{1}{x^3}$	o)	$f(x) = \frac{1}{x^{99}}$

13.5 Determine f'(x):

a)	$f(x) = 3^x$	b)	$f(x) = 5^x$	c)	$f(x) = 18^x$
d)	$f(x) = \left(\frac{2}{3}\right)^x$	e)	$f(x) = \left(\frac{13}{17}\right)^x$	f)	$f(x) = \left(\frac{1}{4}\right)^x$
g)	$f(x) = \left(\frac{1}{e}\right)^x$	h)	$f(x) = \left(\frac{3}{e}\right)^x$	i)	$f(x) = \left(\frac{e}{3}\right)^x$

13.6 Determine the rate of change of the function f at the indicated values of x:

a) $f(x) = x$

i) $x = 0$ ii) $x = 1$ iii) $x = -2$

b) $f(x) = x^5$

i) $x = 0$ ii) $x = 2$ iii) $x = -\frac{2}{3}$

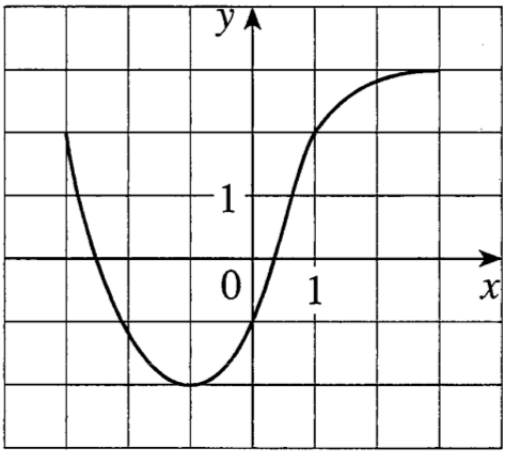
c) $f(x) = x^{-4}$

i) $x = -1$ ii) $x = -\frac{4}{3}$ iii) $x = 0$

d) $f(x) = \left(\frac{2}{3}\right)^x$

i) $x = 0$ ii) $x = 1$ iii) $x = -2$

13.7 The graph of a function f ist given as follows:



Estimate the rate of change $f'(x_0)$ at the given x_0:

a) $x_0 = -1$ b) $x_0 = 0$

c) $x_0 = 1$ d) $x_0 = -2$

Hints:
- Draw the tangent to the graph of f at the given x_0.
- Estimate the slope of the tangent.

13.8 * The rate of change $f'(x_0)$ of f at x_0 can be determined by looking at the secant through the points $A(x_0 \mid f(x_0))$ and $B(x_0+\Delta x \mid f(x_0+\Delta x))$ of the graph of f. The slope of this secant tends towards the slope of the tangent at $A(x_0 \mid f(x_0))$ as Δx tends towards 0.

It has been shown in class how to determine $f'(x_0)$ for the quadratic function $f(x) = x^2$.

Find $f'(x_0)$ for the following functions f:

a) $f(x) = x^3$ b) $f(x) = \frac{1}{x^2}$

Answers

13.1 a) i) ...

ii) $f'(2) = 0$

b) i) ...

ii) $f'(x_0) = 0$

c) i) ...

ii) $f'(4) = 2$

d) i) ...

ii) $f'(x_0) = m$

e) * i) ...

ii) $f'(x_0) = \begin{cases} 1 & (x_0 > 0) \\ -1 & (x_0 < 0) \\ \text{not defined} & (x_0 = 0) \end{cases}$

13.2 a) i) $f': \mathbb{R} \to \mathbb{R}$
$x \to y = f'(x) = 0$

ii) ...

b) i) $f': \mathbb{R} \to \mathbb{R}$
$x \to y = f'(x) = 0$

ii) ...

c) i) $f': \mathbb{R} \to \mathbb{R}$
$x \to y = f'(x) = 2$

ii) ...

d) i) $f': \mathbb{R} \to \mathbb{R}$
$x \to y = f'(x) = m$

ii) ...

e) * i) $f': \mathbb{R}\backslash\{0\} \to \mathbb{R}$
$x \to y = f'(x) = \begin{cases} 1 & (x>0) \\ -1 & (x<0) \end{cases}$

ii) ...

13.3 a) $D = \{x: x \in \mathbb{R} \text{ and } x \geq -1\}$

b) $D_1 = \{x: x \in \mathbb{R} \text{ and } x > -1\}$

Hints:
- The square root of a negative number is not defined.
- Division by zero is not defined.

13.4

a)	$f'(x) = 0$	b)	$f'(x) = 0$	c)	$f'(x) = 0$
d)	$f'(x) = 3x^2$	e)	$f'(x) = 4x^3$	f)	$f'(x) = 5x^4$
g)	$f'(x) = 17x^{16}$	h)	$f'(x) = 200x^{199}$	i)	$f'(x) = 100'001x^{100'000}$
j)	$f'(x) = -x^{-2}$	k)	$f'(x) = -2x^{-3}$	l)	$f'(x) = -17x^{-18}$
m)	$f'(x) = -\frac{1}{x^2}$	n)	$f'(x) = -\frac{3}{x^4}$	o)	$f'(x) = -\frac{99}{x^{100}}$

13.5 a) $f'(x) = 3^x \ln(3)$ b) $f'(x) = 5^x \ln(5)$ c) $f'(x) = 18^x \ln(18)$

d) $f'(x) = \left(\frac{2}{3}\right)^x \ln\left(\frac{2}{3}\right)$ e) $f'(x) = \left(\frac{13}{17}\right)^x \ln\left(\frac{13}{17}\right)$

f) $f'(x) = \left(\frac{1}{4}\right)^x \ln\left(\frac{1}{4}\right) = -\frac{\ln(4)}{4^x}$

Hint:
- Logarithm rules (see formulary) can be applied in order to simplify the result.

g) $f'(x) = -\frac{1}{e^x}$ h) $f'(x) = \left(\frac{3}{e}\right)^x (\ln(3) - 1)$ i) $f'(x) = \left(\frac{e}{3}\right)^x (1 - \ln(3))$

13.6 a) $f'(x) = 1$

i) $f'(0) = 1$ ii) $f'(1) = 1$ iii) $f'(-2) = 1$

b) $f'(x) = 5x^4$

i) $f'(0) = 0$ ii) $f'(2) = 80$ iii) $f'\left(-\frac{2}{3}\right) = \frac{80}{81}$

c) $f'(x) = -\frac{4}{x^5}$

i) $f'(-1) = 4$ ii) $f'\left(-\frac{4}{3}\right) = \frac{243}{256}$ iii) $f'(0)$ is not defined

d) $f'(x) = \left(\frac{2}{3}\right)^x \ln\left(\frac{2}{3}\right)$

i) $f'(0) = \ln\left(\frac{2}{3}\right)$ ii) $f'(1) = \frac{2}{3}\ln\left(\frac{2}{3}\right)$ iii) $f'(-2) = \frac{9}{4}\ln\left(\frac{2}{3}\right)$

13.7 a) $f'(-1) \approx 0$ b) $f'(0) \approx 2$

c) $f'(1) \approx \frac{3}{2}$ d) $f'(-2) \approx -\frac{5}{3}$

13.8 * a) … b) …