Exercises 12 Exponential function and equations Ordinary annuity, annuity due

Objectives

- be able to calculate the present and the future value of an annuity if constant payments are made at the beginning or at the end of each compounding period.
- be able to treat specific annuity tasks.

Problems

Ordinary annuity

12.1 Find the future value of an annuity of $1300 paid at the end of each year for 5 years, if interest is earned at a rate of 6%, compounded annually.

12.2 A family wants to have a $200'000 college fund for their children at the end of 20 years. What contribution must be made at the end of each quarter if their investment pays 7.6%, compounded quarterly?

12.3 If $2500 is deposited at the end of each quarter in an account that earns 5% compounded quarterly, after how many quarters will the account contain $80'000?

12.4 Assume that money on a savings account pays 1.5%, compounded annually. In order to have 20'000 CHF at the end of 10 years, ...

a) ... what payment must be made at the end of each year?

b) ... what amount has to be paid in at the beginning of the ten years if no more payments are made for the rest of the time?

c) Compare the answers in a) and b), and explain why the payment made in b) is a smaller amount than the sum of the 10 payments made in a).

12.5 Two twins are 23 years old and have different investment strategies.

Suppose that twin 1 invests $2000 at the end of each year for 10 years only (until age 33) in an account that earns 8%, compounded annually. Suppose that twin 2 waits until turning 40 to begin investing.

How much must twin 2 put aside at the end of each year for the next 25 years in an account that earns 8%, compounded annually, in order to have the same amount as twin 1 when he turns 65?

Hints:
- It is helpful to draw a diagram which shows the investment strategies of the two twins with respect to time.
- The money twin 1 has paid in by the time he turns 33 pays interest until he turns 65.

12.6 Find the present value of an annuity of $6000 paid at the end of each 6-month period for 8 years if fhe interest rate is 8%, compounded semiannually.

12.7 With a present value of $135'000, what is the size of the withdrawals that can be made at the end of each quarter for the next 10 years if money is worth 6.4%, compounded quarterly?

12.8 A personal account earmarked as a retirement supplement contains $242'000. Suppose $200'000 is used to establish an annuity that earns 6%, compounded quarterly, and pays $4500 at the end of each quarter. How long will it be until the account balance is $0?

12.9 Mr. X owns a caravan. He could sell it now for 20'000 CHF. Alternatively, he could rent it out for 10 years at 2100 CHF per year, the rent being paid at the end of each year. After 10 years the caravan would be completely depreciated. Mr. X could invest the revenues (for either selling or renting out the caravan) at 7%, compounded annually. Which alternative is more beneficial for Mr. X?

12.10 In a building loan contract 3600 CHF are paid in at the end of each year. The money earns interest at an annual rate of 3%. After 10 years twice the saved money is paid out. The debts, worth 5%, compounded annually, have to be paid off within 10 years by instalments due at the end of each year. What is the size of the annual payments (in order to pay off the debts)?

Annuity due

12.11 Find the future value of an annuity due of $100 each quarter for 2.5 years at 12%, compounded quarterly.

12.12 How much must be deposited at the beginning of each year in an account that pays 8%, compounded annually, so that the account will contain $24'000 at the end of 5 years?

12.13 If an account that earns 5%, compounded quarterly, contains $80'000 at the beginning and $2500 is withdrawn at the beginning of each quarter, after how many quarters will the account contain $0?

12.14 What amount must be set aside now to generate payments of $50'000 at the beginning of each year for the next 12 years if money is worth 5.92%, compounded annually?

12.15 A year-end bonus of $25'000 will generate how much money at the beginning of each month for the next year, if it can be invested at 6.48%, compounded monthly?

Miscellaneous problems

12.16 A couple has determined that they need $300'000 to establish an annuity when they retire in 25 years. How much money should they deposit at the end of each month in an investment plan that pays 10%, compounded monthly, so they will have the $300'000 in 25 years?

12.17 Mr. Gordon plans to invest $300 at the end of each month in an account that pays 9%, compounded monthly. After how many months will the account be worth $50'000?

12.18 Grandparents plan to open an account on their grandchild's birthday and contribute each month until she goes to college. How much must they contribute at the beginning of each month in an investement that pays 12%, compounded monthly, if they want the balance to be $180'000 at the end of 18 years?

12.19 An insurance settlement of $750'000 must replace Trixie Eden's income for the next 40 years. What income will this settlement provide at the end of each month if it is invested in an annuity that earns 8.4%, compounded monthly?

12.20 Juanita Domingo's parents want to establish a college trust for her. They want to make 16 quarterly withdrawals of $2000, with the first withdrawal 3 months from now. If money is worth 7.2%, compounded quarterly, how much must be deposited now to provide for this trust?

Answers

12.1 $A_n = p \frac{q^n - 1}{q - 1}$ where p = \$1300, q = 1 + 6% = 1.06, n = 5

$\Rightarrow A_5 = \$7328.22$

12.2 $p = \frac{A_n(q-1)}{q^n - 1}$ where A_n = \$200'000, $q = 1 + \frac{7.6\%}{4}$, n = 80

$\Rightarrow$ p = \$1083.40

12.3 $n = \frac{\log_a\left(\frac{A_n(q-1)}{p} + 1\right)}{\log_a(q)}$ where A_n = \$80'000, p = \$2500, $q = 1 + \frac{5\%}{4}$, a := 10

$\Rightarrow$ n = 27.08... $\rightarrow$ 28 quarters = 7 years

12.4 a) Ordinary annuity

$p = \frac{A_n(q-1)}{q^n - 1}$ where A_n = \$20'000, q = 1 + 1.5%, n = 10

$\Rightarrow$ p = 1868.70 CHF (rounded up)

b) Compound interest

$C_0 = \frac{C_n}{q^n}$ where C_n = \$20'000, q = 1 + 1.5%, n = 10

$\Rightarrow C_0$ = 17'233.35 CHF (rounded up)

c) …

12.5 Twin 1: Ordinary annuity (from age 23 to age 33)

$A_n = p \frac{q^n - 1}{q - 1}$ where p = \$2000, q = 1.08, n = 10

$\Rightarrow A_{10}$ = capital at the age of 33 = \$28'973.12

Compound interest (from age 33 to age 65)

$C_n = C_0 q^n$ where $C_0 = A_{10}$, q = 1.08, n = 32

$\Rightarrow C_{32}$ = capital at the age of 65 = \$340'059.97 (= capital of twin 2 at the age of 65)

Twin 2: Ordinary annuity (from age 40 to age 65)

$p = \frac{A_n(q-1)}{q^n - 1}$ where $A_n = C_{32}$ (twin 1) = \$340'059.97, q = 1.08, n = 25

$\Rightarrow$ p = \$4651.61

12.6 $A_0 = p \frac{q^n - 1}{q^n(q-1)}$ where p = \$6000, $q = 1 + \frac{8\%}{2}$, n = 16 (8 years = 16 half-years)

$\Rightarrow A_0$ = \$69'913.77

12.7 $p = \frac{A_0 q^n(q-1)}{q^n - 1}$ where A_0 = \$135'000, $q = 1 + \frac{6.4\%}{4}$, n = 40 (10 years = 40 quarters)

$\Rightarrow$ p = \$4595.46

12.8 $n = \frac{\log_a\left(\frac{p}{p - A_0(q-1)}\right)}{\log_a(q)}$ where A_0 = \$200'000, p = \$4500, $q = 1 + \frac{6\%}{4}$, a := 10

$\Rightarrow$ n = 73.78... $\rightarrow$ 73 quarters (less than \$4500 at the end of the 74th quarter)

12.9 Alternative 1 (selling the caravan): Compound interest

$C_n = C_0\, q^n$ where C_0 = 20'000 CHF, q = 1.07, n = 10

$\Rightarrow$ C_{10} = 39'343 CHF (rounded)

Alternative 2 (renting out the caravan): Ordinary annuity

$A_n = p\,\frac{q^n - 1}{q - 1}$ where p = 2100 CHF, q = 1.07, n = 10

$\Rightarrow$ A_{10} = 29'015 CHF (rounded)

$\Rightarrow$ $C_{10} > A_{10}$, i.e. alternative 1 is more beneficial

12.10 2 annuities: first 10 years (paying in money), second 10 years (paying off the debts)

- first 10 years (saving money, i.e. paying in money)

$A_n = p\,\frac{q^n - 1}{q - 1}$ where p = 3600 CHF, q = 1.03, n = 10

$\Rightarrow$ A_{10} = 41'270 CHF (rounded)

- second 10 years (paying off debts)

$p = \frac{A_0\, q^n (q-1)}{q^n - 1}$ where A_0 = 41'270 CHF, q = 1.05 , n = 10

$\Rightarrow$ p = 5345 CHF (rounded)

12.11 $A_n = pq\,\frac{q^n - 1}{q - 1}$ where p = \$100, $q = 1 + \frac{12\%}{4}$, n = 10 (2.5 years = 10 quarters)

$\Rightarrow$ A_{10} = \$1180.78

12.12 $p = \frac{A_n(q-1)}{q(q^n - 1)}$ where A_n = \$24'000, q = 1.08, n = 5

$\Rightarrow$ p = \$3787.92

12.13 $n = \frac{\log_a\left(\frac{pq}{pq - A_0(q-1)}\right)}{\log_a(q)}$ where A_0 = \$80'000, p = \$2500, $q = 1 + \frac{5\%}{4}$, a := 10

$\Rightarrow$ n = 40.46... $\rightarrow$ 40 quarters (less than \$2500 at the beginning of the 41st quarter)

12.14 $A_0 = p\,\frac{q^n - 1}{q^{n-1}(q-1)}$ where p = \$50'000, q = 1.0592, n = 12

$\Rightarrow$ A_0 = \$445'962.23

12.15 $p = \frac{A_0\, q^{n-1}(q-1)}{q^n - 1}$ where A_0 = \$25'000, $q = 1 + \frac{6.48\%}{12}$, n = 12 (1 year = 12 months)

$\Rightarrow$ p = \$2145.59

12.16 Ordinary annuity

$p = \frac{A_n(q-1)}{q^n - 1}$ where A_n = \$300'000, $q = 1 + \frac{10\%}{12}$, n = 300 (25 years = 300 months)

$\Rightarrow$ p = \$226.10

12.17 Ordinary annuity

$n = \frac{\log_a\left(\frac{A_n(q-1)}{p} + 1\right)}{\log_a(q)}$ where A_n = \$50'000, p = \$300, $q = 1 + \frac{9\%}{12}$, a := 10

$\Rightarrow$ n = 108.52... $\rightarrow$ 109 months = 9 years 1 month

12.18 Annuity due

$p = \frac{A_n(q-1)}{q(q^n - 1)}$ where A_n = \$180'000, $q = 1 + \frac{12\%}{12}$, n = 216 (18 years = 216 months)

$\Rightarrow$ p = \$235.16

12.19 Ordinary annuity, income = monthly payment p

$p = \frac{A_0\, q^n(q-1)}{q^n - 1}$ where A_0 = \$750'000, $q = 1 + \frac{8.4\%}{12}$, n = 480 (40 years = 480 months)

$\Rightarrow$ p = \$5441.23

12.20 Ordinary annuity

$A_0 = p\,\frac{q^n - 1}{q^n(q-1)}$ where p = \$2000, $q = 1 + \frac{7.2\%}{4}$, n = 16

$\Rightarrow$ A_0 = \$27'590.62