Exercises 10 Exponential function and equations
Exponential equations, logarithm, compound interest

Objectives

- be able to determine simple logarithms without a calculator.
- be able to solve simple exponential equations without a calculator.
- be able to calculate a common logarithm, a natural logarithm with a calculator.
- be able to apply one of the logarithmic properties in order to solve simple exponential equations.
- be able to treat specific compound interest tasks by means of logarithms.

Problems

10.1 Solve the exponential equations below **without** a calculator, i.e. find the solutions by guessing.

a) $2^x = 16$ b) $4^x = 64$ c) $5^x = 1$

d) $\left(\frac{3}{2}\right)^x = \frac{27}{8}$ e) $10^x = 1'000'000$ f) $10^x = 10$

10.2 Determine the following logarithms **without** a calculator.

a) $\log_3(27)$ b) $\log_4(16)$ c) $\log_2(128)$

d) $\log_{10}(1000)$ e) $\log_{10}(1)$

10.3 Determine the logarithms below **with** your calculator.

a) $\log(1.1)$ b) $\ln(1.1)$ c) $\log(9)$

d) $\ln(9)$ e) $\log(2345.67)$ f) $\ln(2345.67)$

10.4 Solve the following exponential equations.

a) $10^x = 21$ b) $10^x = 256.78$ c) $10^x = 1'234'567$

10.5 Solve the exponential equations below.

a) $3^x = 99$ b) $1.01^x = 1.5$ c) $3^{x+4} = 5$

d) $5^{2x-1} = 12$ e) $0.2^{x-3} = 27$ f) $1 - e^{5x} = 0.3$

10.6 An initial capital C_0 is invested at an interest rate r, compounded annually. After n years the capital amounts to C_n. Determine n.

a) $C_0 = 1000$ CHF r = 1.00% $C_n = 1220$ CHF (rounded)

b) $C_0 = 100'000$ CHF r = 2.25% $C_n = 243'519$ CHF (rounded)

10.7 How long would 10'000 CHF have to be invested at 2.5%, compounded annually, to amount to 12'000 CHF?

10.8 How long would any initial capital have to be invested at 1.25%, compounded annually, to double its value?

10.9 An initial capital of 10'000.00 CHF is invested at an unknown interest rate, compounded annually. After 10 years the capital amounts to 11'894.40 CHF. After how many years (from the beginning of the investment) will the capital be worth 15'000.00 CHF?

10.10 The sales decay for a product is given by

$$S = 50'000\ e^{-0.8\,x}$$

where S is the monthly sales and x is the number of months that have passed since the end of a promotional campaign.

a) What will be the sales 4 months after the end of the campaign?

b) How many months after the end of the campaign will sales drop below 1000, if no new campaign is initiated?

10.11 The demand function for a certain commodity is given by

$$p = 100\ e^{-\,q/2}$$

If the price is $1.83 per unit, how many units will be demanded, to the nearest unit?

10.12 On a college campus of 10'000 students, a single student returned to campus infected by a disease. The spread of the disease through the student body is given by

$$y = \frac{10'000}{1 + 9999\ e^{-\,0.99\,t}}$$

where y is the total number infected at time t (in days).

a) How many are infected after 4 days?

b) The school will shut down if 50% of the students are ill. During what day will it close?

10.13 Pollution levels in Lake Erie have been modelled by the equation

$$x = 0.05 + 0.18\ e^{-\,0.38\,t}$$

where x is the volume of pollutants (in cubic kilometers) and t is the time (in years).

a) Find the initial pollution level, i.e. find x when t = 0.

b) How long will it take the pollution level to reach 30% of the initial level?

10.14 * Suppose the supply of x units of a product at price p dollars per unit is given by

$$p = 10 + 5\ \ln(3x + 1)$$

How many units would be supplied when the price is $50 each?

Answers

10.1 a) $x = 4$ b) $x = 3$ c) $x = 0$

d) $x = 3$ e) $x = 6$ f) $x = 1$

10.2 a) 3

Hint:
- The expression $\log_3(27)$ is the answer to the question "3 to what power is equal to 27?"

b) 2 c) 7

d) 3 e) 0

10.3 a) 0.041... b) 0.095... c) 0.954...

d) 2.197... e) 3.370... f) 7.760...

10.4 a) $x = \log(21) = 1.322...$

Hints:
- Apply log(…) to both sides of the equation.
- Use the fact that $\log(10^x) = x$ for any $x \in \mathbb{R}$.

b) $x = \log(256.78) = 2.409...$

c) $x = \log(1'234'567) = 6.091...$

10.5 a) $x = 4.182...$ b) $x = 40.748...$

c) $x = - 2.535...$

Hint:
- First solve the equation for $x+4$.

d) $x = 1.271…$ e) $x = 0.952…$

f) $x = - 0.071…$

Hints:
- First solve for e^{5x}.
- Then solve for $5x$.

10.6 $n = \frac{\log\left(\frac{C_n}{C_0}\right)}{\log(q)}$

a) $n = 20$ b) $n = 40$

10.7 $n = \frac{\log\left(\frac{C_n}{C_0}\right)}{\log(q)} = \frac{\log\left(\frac{12'000}{10'000}\right)}{\log(1.025)} = 7.38... \rightarrow$ 8 years

10.8 $C_n = C_0 \cdot q^n$
$C_n = 2 \cdot C_0$

$\Rightarrow$ $n = \frac{\log(2)}{\log(1.0125)} = 55.79... \rightarrow$ 56 years

10.9 $r = 1.75\%$
$C_n = 15'000$ CHF for $n = 23.37... \rightarrow$ 24 years

Hints:
- First determine the interest rate r by looking at the first 10 years (C_0 and C_{10} are known, r is unknown).
- Then determine n (C_0, C_n, and r are known, n is unknown).

10.10 a) $S(4) = 2038$

b) $x = 4.9$, i.e. after 4.9 months

Hints:
- Determine x such that $S = 1000$.
- The equation $1000 = 50'000\, e^{-0.8\,x}$ has to be solved for x.
- Use the fact that $\ln(e^x) = x$ for any $x \in \mathbb{R}$.

10.11 $q = 8.0017\ldots \rightarrow$ 8 units

Hint:
- Use the same procedure as in 10.10 b).

10.12 a) $y(4) = 52.18\ldots \rightarrow$ 52 students

b) $t = 9.30\ldots \rightarrow$ the 10th day

Hint:
- The following equation has to be solved for t: $5000 = \frac{10'000}{1 + 9999\, e^{-0.99\,t}}$

10.13 a) $x = 0.23$ km^3

b) $t = 5.91\ldots \rightarrow$ 5.9 years

Hint:
- Use the same procedure as in 10.12 b).

10.14 * $x = 993.31\ldots \rightarrow$ 993 units