Review exercises 2 Differential calculus, integral calculus

Problems

- R2.1 Decide whether the following statements are true or false:
 - a) "The derivative of a function is a function."
 - b) "The derivative of a function at a particular value of the variable is a number."
 - "The function f has a relative maximum at $x = x_1$ if $f'(x_1) = 0$ and $f''(x_1) > 0$."
 - d) "If $f''(x_2) = 0$ and $f'''(x_2) < 0$, then the function f has a point of inflection at $x = x_2$."
 - e) "Suppose that the function f has a relative maximum at $x = x_1$. If there are no other relative maxima and if there is no relative minimum at all, then the relative maximum is the absolute maximum of f."
 - f) "If g' = f, then g is an antiderivative of f."
 - g) "f with f(x) = 2x + 20 is an antiderivative of g with $g(x) = x^2$."
 - h) "f with f(x) = 3x has infinitely many antiderivatives."
 - i) "The indefinite integral of a function is a set of functions."
- R2.2 Determine the value $f(x_0)$, the first derivative $f'(x_0)$, and the second derivative $f''(x_0)$ at x_0 for the following functions f:
 - a) $f(x) = 4x^2(x^2 1)$
 - i) $x_0 = 0$
- ii) $x_0 = -$
- b) $f(x) = (-3x^2 + 2x 1) \cdot e^x$
 - i) $x_0 = 0$
- ii) $x_0 = -1$
- c) $f(x) = (x^2 + 2) \cdot e^{-3x}$
 - i) $x_0 = 1$
- ii) $x_0 = -\frac{1}{3}$
- R2.3 For the given cost function C(x) and revenue function R(x) determine ...
 - i) ... the marginal cost function C'(x).
 - ii) ... the marginal revenue function R'(x).
 - iii) ... the marginal profit function P'(x).
 - a) C(x) = 200 + 40x
- R(x) = 60x
- b) $C(x) = 100 + 20x + 5x^2$
- $R(x) = 100x 2x^2$
- c) $C(x) = 50 + 20x^2 + 3e^{4x}$
- $R(x) = 200x e^{-4x^2}$
- R2.4 For each function, find ...
 - i) ... the relative maxima and minima.
 - ii) ... the points of inflection.
 - a) $f(x) = 2x^3 9x^2 + 12x 1$
 - b) f(x) as in R2.2 a)

R2.5 The total revenue function for a commodity is given by

$$R(x) = 36x - 0.01x^2$$

Find the maximum revenue ...

- a) ... if production is not limited to a certain number of units.
- b) ... if production is limited to at most 1500 units.
- R2.6 If the total cost function for a product is

$$C(x) = 100 + x^2$$

producing how many units x will result in a minimum average cost per unit? Find the minimum average cost.

R2.7 A firm can produce only 1000 units per month. The monthly total cost ist given by

$$C(x) = 300 + 200x$$

dollars, where x is the number produced. If the total revenue is given by

$$R(x) = 250x - \frac{1}{100}x^2$$

dollars, how many items should the firm produce for maximum profit? Find the maximum profit.

- R2.8 Determine the indefinite integrals below:
 - a) $\int (x^4 3x^3 6) dx$
 - b) $\int \left(\frac{1}{2}x^6 \frac{2}{3x^4}\right) dx$
- R2.9 The equation of the third derivative f'' of a function f is given as follows:

$$f'''(x) = 3x + 1$$

Find the equation of the function f such that f''(0) = 0, f'(0) = 1, f(0) = 2

- R2.10 If the marginal cost (in dollars) for producing a product is C'(x) = 5x + 10, with a fixed cost of \$800, what will be the cost of producing 20 units?
- R2.11 A certain firm's marginal cost C'(x) and the derivative of the average revenue $\overline{R}'(x)$ are given as follows:

$$C'(x) = 6x + 60$$

$$\overline{R}'(x) = -1$$

The total cost and revenue of the production of 10 items are \$1000 and \$1700, respectively.

How many units will result in a maximum profit? Find the maximum profit.

R2.12 The demand function for a product is

$$p = f(x) = 49 - x^2$$

and the supply function is

$$p = g(x) = 4x + 4$$

Find the equilibrium point and both the consumer's and the producer's surplus there.

R2.13 The demand function for a product is

$$p = f(x) = 110 - ax^2$$

and the supply function is

$$p = g(x) = 2 - \frac{6}{5}x + bx^2$$

with unknown parameters a and b. The equilibrium price is \$10, and the producer's surplus is \$73.33 Determine the two unknown parameters a and b.

Answers

R2.1 a) true b) true c) false

d) true e) true f) true

g) false h) true i) true

R2.2 a) $f'(x) = 16x^3 - 8x$ $f''(x) = 48x^2 - 8$

i) f(0) = 0 f'(0) = 0 f''(0) = -8

ii) f(-1) = 0 f'(-1) = -8 f''(-1) = 40

b) $f'(x) = (-3x^2 - 4x + 1) \cdot e^x$ $f''(x) = (-3x^2 - 10x - 3) \cdot e^x$

i) f(0) = -1 f'(0) = 1 f''(0) = -3

ii) $f(-2) = -17 \cdot e^{-2} = -2.300...$ $f'(-2) = -3 \cdot e^{-2} = -0.406...$ $f''(-2) = 5 \cdot e^{-2} = 0.676...$

c) $f'(x) = (-3x^2 + 2x - 6) \cdot e^{-3x}$ $f''(x) = (9x^2 - 12x + 20) \cdot e^{-3x}$

> i) $f(1) = 3 \cdot e^{-3} = 0.149...$ $f'(1) = -7 \cdot e^{-3} = -0.348...$ $f''(1) = 17 \cdot e^{-3} = 0.846...$

ii) $f\left(-\frac{1}{3}\right) = \frac{19}{9}e = 5.738...$ $f'\left(-\frac{1}{3}\right) = -7e = -19.027...$

 $f''(-\frac{1}{3}) = 25e = 67.957...$

R2.3 a) i) C'(x) = 40 ii) R'(x) = 60

iii) P'(x) = 20

b) i) C'(x) = 20 + 10x ii) R'(x) = 100 - 4x

iii) P'(x) = 80 - 14x

c) i) $C'(x) = 40x + 12e^{4x}$ ii) $R'(x) = 200 + 8x e^{-4x^2}$

iii) $P'(x) = 200 - 40x - 12e^{4x} + 8x e^{-4x^2}$

R2.4 a) $f(x) = 2x^3 - 9x^2 + 12x - 1$ $f'(x) = 6x^2 - 18x + 12$ f''(x) = 12x - 18

f'''(x) = 12

i) f'(x) = 0 at $x_1 = 1$ and $x_2 = 2$ $f''(x_1) = -6 < 0$ \Rightarrow relative maximum at $x_1 = 1$

 $f''(x_2) = 6 > 0$ \Rightarrow relative minimum at $x_2 = 2$

ii)
$$f''(x) = 0 \text{ at } x_3 = \frac{3}{2}$$

$$f'''(x_3) = 12 \neq 0 \qquad \Rightarrow \qquad \text{point of inflection at } x_3 = \frac{3}{2}$$

b)
$$f(x) = 4x^{2}(x^{2} - 1)$$

$$f'(x) = 16x^{3} - 8x = 8x(2x^{2} - 1)$$

$$f''(x) = 48x^{2} - 8 = 8(6x^{2} - 1)$$

$$f'''(x) = 96x$$

$$\begin{array}{lll} \text{i)} & & f'(x)=0 \text{ at } x_1=0,\, x_2=\frac{1}{\sqrt{2}}\,, \text{ and } x_3=-\frac{1}{\sqrt{2}}\\ & & f''(x_1)=-\,8<0 & \Rightarrow & \text{relative maximum at } x_1=0\\ & & f''(x_2)=16>0 & \Rightarrow & \text{relative minimum at } x_2=\frac{1}{\sqrt{2}}\\ & & f''(x_3)=16>0 & \Rightarrow & \text{relative minimum at } x_3=-\frac{1}{\sqrt{2}} \end{array}$$

ii)
$$f''(x) = 0 \text{ at } x_3 = \frac{1}{\sqrt{6}}$$

$$f'''(x_3) = \frac{96}{\sqrt{6}} \neq 0 \qquad \Rightarrow \qquad \text{point of inflection at } x_3 = \frac{1}{\sqrt{6}}$$

R2.5 a) **Relative** maximum at
$$x_1 = 1800$$

 $R(x_1) = $32'400$

 $R(x) < R(x_1)$ if $x \ne x_1$ as there is no relative minimum

 \Rightarrow R = \$32'400 is the **absolute** maximum revenue at x = 1800.

Relative maximum at x = 1800 lies outside the possible interval $0 \le x \le 1500$ b) R(1500) = \$31'500 > R(0) = 0\$ \Rightarrow R = \$31'500 is the **absolute** maximum revenue at x = 1500.

R2.6
$$\overline{C}(x) = \frac{C(x)}{x} = \frac{100}{x} + x$$

$$\begin{split} \overline{C}(x) &= \frac{C(x)}{x} = \frac{100}{x} + x \\ \overline{C}(x) \text{ has a } \textbf{relative} \text{ minimum at } x_1 = 10 \end{split}$$

 $\overline{C}(20) = \$20$

 $\overline{C}(x) > \overline{C}(x_1)$ if $x \neq x_1$ as there is no relative maximum

 $\Rightarrow \overline{C} = \20 is the **absolute** minimum average cost at x = 10.

R2.7
$$P(x) = R(x) - C(x) = -\frac{1}{100}x^2 + 50x - 300$$

P(x) has a **relative** maximum at $x_1 = 2500$. This is outside the possible interval $0 \le x \le 1000$

P(1000) = \$39'700 > P(0) = -300\$

 \Rightarrow P = \$39'700 is the **absolute** maximum profit at the endpoint x = 1000.

R2.8 a)
$$\int (x^4 - 3x^3 - 6) dx = \frac{x^5}{5} + \frac{3x^4}{4} - 6x + C$$

b)
$$\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right) dx = \frac{x^7}{14} + \frac{2}{9x^3} + C$$

R2.9
$$f(x) = \frac{x^4}{8} + \frac{x^3}{6} + x + 2$$

R2.10
$$C(20) = $2000$$

Hint:

- First, determine the cost function $C(x) \Rightarrow C(x) = \frac{5}{2}x^2 + 10x + 800$

R2.11 P = \$800 is the absolute maximum profit at x = 15 units.

Hints:

- Determine the cost function $C(x) \Rightarrow C(x) = 3x^2 + 60x + 100$
- Determine the average revenue function $\overline{R}(x) \Rightarrow \overline{R}(x) = -x + C$
- Determine the revenue function $R(x) \Rightarrow R(x) = -x^2 + 180x$ Find the profit function $P(x) \Rightarrow P(x) = -4x^2 + 120x 100$
- Find the relative maximum of the profit function P(x).
- Check if the relative maximum is the absolute maximum.
- R2.12 Equilibrium quantity x = 5

Equilibrium price p = 24Consumer's surplus CS = 83.33Producer's surplus PS = 50

R2.13 a = 1b = 0.2