# Exercises 4 Linear function and equations Linear function, simple interest, cost, revenue, profit, break-even

#### Objectives

- be able to think of a relation between two quantities as a function.
- be able to determine the domain, the codomain, the range of a given function.
- be able to draw the graph of a given linear function.
- be able to determine slope and intercept of a linear function.
- know some examples of linear functions in economic and everyday life applications.
- know and understand the term "simple interest".
- be able to perform simple interest calculation.
- know and understand the terms "fixed costs", "variable costs", "total cost", "total revenue", "total profit" and "break-even value".
- be able to apply the concept of linear functions to a new problem.

### Problems

4.1 A taxi driver charges the following fare:

8.00 CHF plus 1.50 CHF per kilometer

Think of the taxi fare as a function f.

- a) Determine the domain D, the codomain B, and the range E of the function.
- b) Draw the graph of the function f.
- 4.2 The taxi fare as described in problem 4.1 can be thought of as a linear function which assigns a fare y (in CHF) to each distance x (in km):

f:  $\mathbb{R}^+ \to \mathbb{R}^+$  $x \to y = f(x) = ax + b$ 

Determine the values of a and b.

- 4.3 Find at least two more examples of linear functions in economics or in an everyday life context.
- 4.4 Graph the linear functions below, and determine both slope and intercept:
  - a) f:  $\mathbb{R} \to \mathbb{R}$  $x \to y = f(x) = -2$
  - b) f:  $\mathbb{R} \to \mathbb{R}$ 
    - $x \rightarrow y = f(x) = 3x 4$
  - c) f:  $\mathbb{R} \to \mathbb{R}$  $x \to y = f(x) = -x + 3$
- 4.5 Simple interest at a rate of 0.5% is paid on an initial bank balance of 5000 CHF.
  - a) Determine the interest that is paid each year.
  - b) Determine the balance after ten years' time.
  - c) Determine both slope and intercept of the corresponding linear function.

4.6 In general, if an initial capital  $C_0$  pays simple interest at an annual rate r (e.g. r = 1.5% = 0.015), the capital  $C_n$  after n years is given by the formula below (see formulary):

 $C_n = C_0(1 + nr)$ 

- a) Verify that the given formula is correct.
- b) Determine both slope and intercept of the corresponding linear function.
- 4.7 An initial capital  $C_0 = 1200$  CHF pays simple interest at an annual interest rate of 1.5%.
  - a) After how many years will the capital exceed 2000 CHF?
  - b) At what annual interest rate (rounded to 0.05%) would the capital exceed 2000 CHF after 20 years' time?

Hint:

- Use the formula given in problem 4.6 and solve it for n and r respectively.

4.8 A mobile phone company offers three different tariffs:

Tariff A:	monthly basic fee of 10 CHF plus 0.20 CHF per minute
Tariff B:	monthly basic fee of 25 CHF plus 0.10 CHF per minute
Tariff C:	no basic fee, 0.60 CHF per minute

Think of the three tariffs as linear functions.

- a) Draw the graphs of the three functions in one common coordinate system.
- b) Determine the total fee for each tariff for a monthly phone call duration of 1 hour.
- c) For what monthly phone call duration tariff A is cheaper than tariff C?
- d) For what monthly phone call duration tariff B is cheaper than tariff A?
- 4.9 (from: Bittinger, Ellenbogen: Calculus and its applications, Pearson 2007, ISBN 0-321-48543-2)

**EXAMPLE 9** Business: Total Cost. Raggs, Ltd., a clothing firm, has **fixed costs** of \$10,000 per year. These costs, such as rent, maintenance, and so on, must be paid no matter how much the company produces. To produce x units of a certain kind of suit, it costs \$20 per suit (unit) in addition to the fixed costs. That is, the **variable costs** for producing x of these suits are 20x dollars. These costs are due to the amount produced and stem from items such as material. wages, fuel, and so on. The **total cost** C(x) of producing x suits in a year is given by a function C:

C(x) = (Variable costs) + (Fixed costs) = 20x + 10,000.

- a) Graph the variable-cost, the fixed-cost, and the total-cost functions.
- b) What is the total cost of producing 100 suits? 400 suits?

#### 4.10 (see next page)

4.10 (from: Bittinger, Ellenbogen: Calculus and its applications, Pearson 2007, ISBN 0-321-48543-2)

**EXAMPLE 10** Business: Profit-and-Loss Analysis. When a business sells an item, it receives the *price* paid by the consumer (this is normally greater than the *cost* to the business of producing the item).

**a)** The **total revenue** that a business receives is the product of the number of items sold and the price paid per item. Thus, if Raggs, Ltd., sells *x* suits at \$80 per suit, the total revenue R(x), in dollars, is given by

R(x) = Unit price  $\cdot$  Quantity sold = 80x.

If C(x) = 20x + 10,000 (see Example 9), graph R and C using the same set of axes.

**b)** The **total profit** that a business receives is the amount left after all costs have been subtracted from the total revenue. Thus, if P(x) represents the total profit when *x* items are produced and sold, we have

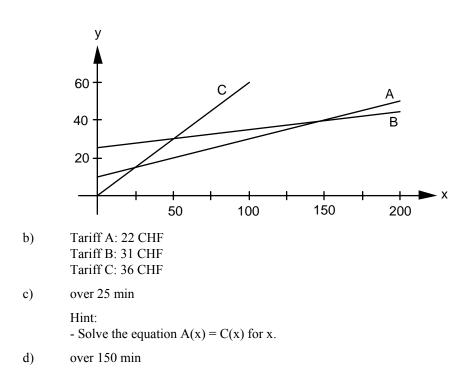
P(x) = (Total revenue) - (Total costs) = R(x) - C(x).

Determine P(x) and draw its graph using the same set of axes as was used for the graph in part (a).

c) The company will *break even* at that value of x for which P(x) = 0 (that is, no profit and no loss). This is the point at which R(x) = C(x). Find the **break-even** value of x.

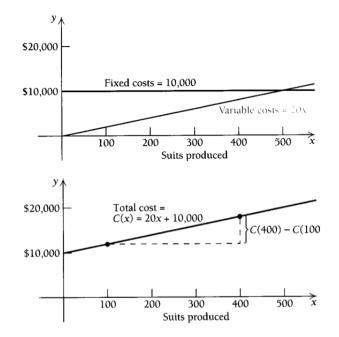
## Answers

4.1	a)	$D = \mathbb{R}^+ \text{ (distance/km)}$ $B = \mathbb{R}^+ \text{ (fare/CHF)}$ $E = \{y: y \in \mathbb{R}^+ \text{ and } y > 8\} \text{ or } E = (8, \infty)$	
	b)		
4.2	a = 1.	1.5, b = 8	
4.3			
4.4	a)	Slope $a = 0$ , intercept $b = -2$	
	b)	Slope $a = 3$ , intercept $b = -4$	
	c)	Slope $a = -1$ , intercept $b = 3$	
4.5	a)	25 CHF	
	b)	5250 CHF	
	c)	f: $\mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ x $\rightarrow y = f(x) = ax + b$	
		Slope $a = 25$ , intercept $b = 5000$	
4.6	a)	Interest paid each year = $r \cdot C_0$ Capital $C_n$ after n years = $C_0 + n \cdot (r \cdot C_0) = C_0(1 + nr)$	
	b)	Slope $a = r \cdot C_0$ , intercept $b = C_0$	
		Hints: - Compare the formula $C_n = C_0(1 + nr)$ with the general form of the equation of a linear function. - $C_n = C_0(1 + nr) = an + b = f(n)$	
4.7	a)	$n = \frac{\frac{C_n}{C_0} - 1}{r}$ where $C_0 = 1200$ CHF, $C_n = 2000$ CHF, $r = 1.5\% = 0.015$	
		$\Rightarrow n = 44.4 \rightarrow 45 \text{ years}$ $r = \frac{\frac{C_n}{C_0} - 1}{\frac{1}{C_0}} \qquad \text{where } C_0 = 1200 \text{ CHF}, C_n = 2000 \text{ CHF}$	
	b)	$r = \frac{C_0}{n}$ where $C_0 = 1200$ CHF, $C_n = 2000$ CHF	
		$n = 20 \implies r = 0.03333 = 3.333\%$ $n = 19 \implies r = 0.03508 = 3.508\%$	
		$\Rightarrow$ r $\in \{3.35\%, 3.40\%, 3.45\%, 3.50\%\}$	
4.8	a)	Tariff A: A: $\mathbb{R}_{0^+} \rightarrow \mathbb{R}_{0^+}$ $x \rightarrow y = A(x) = 0.2 x + 10$	
		Tariff B: B: $\mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ $x \rightarrow y = B(x) = 0.1 x + 25$	
		Tariff C: C: $\mathbb{R}_0^+ \to \mathbb{R}_0^+$ $x \to y = C(x) = 0.6 x$ Direct proportionality: fee y is direct proportional to phone call duration x.	



4.9

a) The variable-cost and fixed-cost functions appear in the graph on the left below. The total-cost function is shown in the graph on the right. From a practical standpoint, the domains of these functions are nonnegative integers 0, 1, 2, 3, and so on, since it does not make sense to make either a negative number or a fractional number of suits. It is common practice to draw the graphs as though the domains were the entire set of nonnegative real numbers.



**b)** The total cost of producing 100 suits is

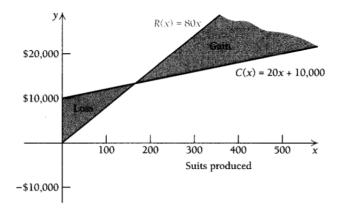
$$C(100) = 20 \cdot 100 + 10,000 = $12,000.$$

The total cost of producing 400 suits is

$$C(400) = 20 \cdot 400 + 10,000$$
  
= \$18,000.

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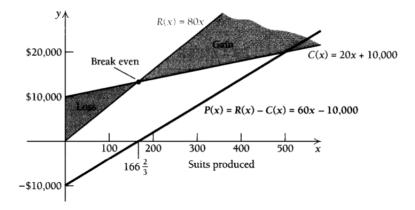
- 4.10
- **a)** The graphs of R(x) = 80x and C(x) = 20x + 10,000 are shown below. When C(x) is above R(x), a loss will occur. This is shown by the region shaded red. When R(x) is above C(x), a gain will occur. This is shown by the region shaded gray.



**b)** To find *P*, the profit function, we have

$$P(x) = R(x) - C(x) = 80x - (20x + 10,000)$$
  
= 60x - 10,000.

The graph of P(x) is shown by the heavy line. The red portion of the line shows a "negative" profit, or loss. The black portion of the heavy line shows a "positive" profit, or gain.



- **c)** To find the break-even value, we solve R(x) = C(x):
  - R(x) = C(x) 80x = 20x + 10,000 60x = 10,000 $x = 166\frac{2}{3}.$

How do we interpret the fractional answer, since it is not possible to produce  $\frac{2}{3}$  of a suit? We simply round to 167. Estimates of break-even values are usually sufficient since companies want to operate well away from break-even values in order to maximize profit.