Review exercises 1 Functions and equations

Problems

R1.1 Which of the following relations are functions? Explain your answers.

a) $f_1 : \mathbb{R}_0^+ \to \mathbb{R}^+, x \to y = f_1(x) = \sqrt{x}$

b) $f_2 : \{2, 3, 4, ...\} \to \mathbb{N}, x \to y = f_2(x) = x - 1$

c) D = Set of all Swiss cantons
B = Set of all Swiss towns and cities
$f_3 : D \to B, x \to y = f_3(x)$ = capital of x

d) $f_4 : \{x: x \in \mathbb{R} \text{ and } x \geq 3\} \to \mathbb{R}, x \to y = f_4(x) = \frac{1}{x^2 - 9}$

e) $f_5 : \mathbb{R}_0^+ \to \mathbb{R}, x \to y = f_5(x) = \log_a(x)$

R1.2 Determine the largest possible domain D and the corresponding range E of the following functions:

a) $f_1 : D \to \mathbb{R}, x \to y = f_1(x) = \sqrt{9 - x}$

b) $f_2 : D \to \mathbb{R}, x \to y = f_2(x) = \frac{1}{1 - x^2}$

R1.3 If $f(x) = 9x - x^2$, find ...

a) ... f(0). b) ... f(-3).

c) ... $\frac{f(x + h) - f(x)}{h}$ and simplify the expression.

R1.4 Solve the equations below:

a) $3x - 8 = 23$

b) $\frac{6}{3x - 5} = \frac{6}{2x + 3}$

c) $\frac{2x + 5}{x + 7} = \frac{1}{3} + \frac{x - 11}{2x + 14}$

R1.5 Solve the following equations for x, and determine the solution sets.
Take into account that the parameters a and p can be any real numbers.

a) $ax = 60$

b) $(p - 1)px = p^2 - 1$

R1.6 Solve each system of equations:

a) $2x + y = 19$
$x - 2y = 12$

b) $6x + 3y = 1$
$y = - 2x + 1$

R1.7 Find the equation of the linear function whose graph ...

a) ... has slope 4 and intercept 2.

b) ... passes through (-2|1) and has slope $\frac{2}{5}$.

c) ... passes through (-2|7) and (6|-4).

d) ... passes through (1|6) and is parallel to y = 4x - 6.

R1.8 A certain product has the following supply and demand functions:

$p = f_s(q) = (4q + 5)$ CHF
$p = f_d(q) = (-2q + 81)$ CHF

a) If the price is $53, how many units are supplied and how many are demanded?

b) Find both the equilibrium quantity and the equilibrium price.

R1.9 The total cost and total revenue for a certain product are given by the following:

C(x) = (38.80x + 4500) CHF
R(x) = 61.30x CHF

a) Determine the fixed costs.

b) Determine the variable costs for producing 10 units.

c) Find the number of units required to break even.

R1.10 The supply function and the demand function for a product are linear and are determined by the tables that follow. Find the quantity and price that will give market equilibrium.

Supply function		Demand function	
Price	Quantity	Price	Quantity
$100	200	$200	200
$200	400	$100	400
$300	600	$0	600

R1.11 Find the solutions to each equation:

a) $4x - 3x^2 = 0$

b) $3x^2 - 6x = 9$

c) $4x^2 + 25 = 0$

d) $\frac{1}{x} + 2x = \frac{1}{3} + \frac{x+1}{x}$

e) $\frac{x-4}{x-5} = \frac{30-x^2}{x^2-5x}$

R1.12 Find the equation of the quadratic function whose graph ...

a) ... has the vertex (2|4) and passes through (3|3).

b) ... passes through (-3|-3), (0|3), and (3|0).

R1.13 The supply function for a product is given by $p = q^2 + 300$, and the demand is given by $p + q = 410$. Find the equilibrium quantity and price.

R1.14 If total costs for a product are given by $C(x) = 1760 + 8x + 0.6x^2$ and total revenues are given by $R(x) = 100x - 0.4x^2$, find the break-even points.

R1.15 Consider the following function f:

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \to y = f(x) = (k - x)(x - 2) - k(x^2 - 2) - 1 \quad (k \in \mathbb{R})$$

Determine the value(s) of k such that the graph of f and the x-axis have exactly one point in common.

R1.16 The functions f and g are defined as follows:

$$f: \mathbb{R} \to \mathbb{R}, x \to y = f(x) = 2x^2 + 4x + 1$$

$$g: \mathbb{R} \to \mathbb{R}, x \to y = g(x) = ax + \frac{1}{2} \quad (a \in \mathbb{R}\backslash\{0\})$$

Determine the value(s) of a such that the graphs of f and g touch.

R1.17 Find the equation of the exponential function whose graph passes through P and Q.

a) P(0|1) Q(2|9)

b) P(1|20) Q(2|100)

R1.18 Evaluate each logarithm without using a calculator:

a) $\log_5(1)$

b) $\log_2(8)$

c) $\log_3\left(\frac{1}{3}\right)$

d) $\log_3(3^8)$

e) $e^{\ln(5)}$

f) $10^{\log(3.15)}$

R1.19 Solve each equation:

a) $6^{4x} = 46'656$

b) $8000 = 250 \cdot 1.07^x$

c) $312 = 300 + 300\, e^{-0.08x}$

R1.20 If $8000 is borrowed at 12% simple interest for 3 years, what is the future value of the loan at the end of the 3 years?

R1.21 Mary Toy borrowed $2000 from her parents and repaid them $2100 after 9 months. What simple interest rate did she pay?

R1.22 How much summer earnings must a college student deposit on August 31 in order to have \$3000 for tuition and fees on December 31 of the same year, if the investement earns 6% simple interest?

R1.23 If \$1000 is invested for 4 years at 8%, compounded quarterly, how much interest will be earned?

R1.24 How much must one invest now in order to have \$18'000 in 4 years if the investement earns 5.4%, compounded monthly?

R1.25 In 1990 an African country had a population of 4.5 million. The population has been increasing at 4% per year. What will the population be in 2010 if the growth factor does not change?

R1.26 A company wants to have \$250'000 available in 4 1/2 years for new construction. How much must be deposited at the beginning of each quarter to reach this goal if the investement earns 10.2%, compounded quarterly?

R1.27 A retirement account that earns 6.8%, compounded semiannually, contains \$488'000. How long can \$40'000 be withdrawn at the end of each half-year until the account balance is \$0?

R1.28 Three years from now, a couple plan to spend 4 months travelling in China, Japan, and Southeast Asia. When they take their trip, they would like to withdraw \$5000 at the beginning of each month to cover their expenses for that month. Starting now, how much must they deposit at the beginning of each month for the next 3 years so that the account will provide the money they want while they are travelling? Assume that such an account pays 6.6%, compounded monthly.

R1.29 Mr. S is obligated to pay 25'000 CHF at the end of each of the following 8 years to his divorced wife. As a result of a personal profit in his company, he is able to pay the whole sum at the end of the first year (instead of making 8 payments at the end of each year). What amount of money does he have to pay at the end of the first year if the annual interest rate has been fixed at 4.5%?

R1.30 Mr. P is thinking about an investement for his retirement. He would like to withdraw 8000 CHF from an account at the end of each year for 15 years starting at the end of the year in which he turns 60. He assumes an annual interest rate of 2.5%.

a) He wants to save the money by making 30 constant payments at the end of each year until turning 55. How much must he pay in each year, if his banks pays him 3%, compounded annually?

b) Mr. P has won 40'000 CHF in a lottery! Would this amount be sufficient for his retirement scheme if he pays the money in at the end of the year in which he turns 25? Assume the same interest rate as in a).

Answers

R1.1 a) no function (no element in $\mathbb{R}^+$ is assigned to x = 0)

b) function

c) function

d) no function (f not defined for x = 3)

e) no function (f not defined for x = 0)

Hints:
- A function must be defined for each element of the domain.
- A function must be unique, i.e. only one element of the codomain is assigned to each element in the domain.

R1.2 a) $D = \{x: x \in \mathbb{R} \text{ and } x \leq 9\}$
$E = \mathbb{R}_0^+$

Hints:
- D: The square root of a negative number is not defined, i.e. $9 - x \geq 0$
- E: Starting from x = 9, $y = f_1(9) = 0$, the values of y increase as the values of x decrease.

b) $D = \mathbb{R} \setminus \{-1, 1\}$
$E = \mathbb{R} \setminus \{y: y \in \mathbb{R} \text{ and } 0 \leq y < 1\}$

Hint:
- D: Division by 0 is not defined, i.e. $1 - x^2 \neq 0$

R1.3 a) f(0) = 0

b) f(-3) = -36

c) $f(x+h) = 9(x+h) - (x+h)^2$
$\frac{f(x+h) - f(x)}{h} = 9 - 2x - h$

R1.4 a) $S = \left\{\frac{31}{3}\right\}$

b) S = {8}

Hint:
- First get rid of the fraction by multiplying by the least common denominator .

c) S = {-7}

Hint:
- Use the same procedure as in b).

R1.5 a) dividing by a only allowed if $a \neq 0$

if $a = 0$: $0 = 60$ (false for each $x \in \mathbb{R}$) $\Rightarrow$ $S = \{ \}$
if $a \neq 0$: $x = \frac{60}{a}$ $\Rightarrow$ $S = \left\{\frac{60}{a}\right\}$

b) dividing by p only allowed if $p \neq 0$
dividing by (p - 1) only allowed if $(p - 1) \neq 0$, i.e. $p \neq 1$

if $p = 0$: $0 = -1$ (false for each $x \in \mathbb{R}$) $\Rightarrow$ $S = \{ \}$
if $p = 1$: $0 = 0$ (true for each $x \in \mathbb{R}$) $\Rightarrow$ $S = \mathbb{R}$
if $p \neq 0$ and $p \neq 1$: $x = \frac{p+1}{p}$ $\Rightarrow$ $S = \left\{\frac{p+1}{p}\right\}$

Hints:
- Divison by 0 is not defined.
- A division by a number that contains the parameter a or p requires a case differentiation.

R1.6 a) (x, y) = (10, -1)
S = {(10, -1)}

b) no solution
S = { }

Hints:
- First solve one equation for y (or x).
- Substitute the expression for y (or x) in the other equation.
- Solve the equation for x (or y).

R1.7 a) $y = f(x) = 4x + 2$

b) $y = f(x) = \frac{2}{5}x + \frac{9}{5}$

c) $y = f(x) = -\frac{11}{8}x + \frac{17}{4}$

d) $y = f(x) = 4x + 2$

Hints:
- First state the general form of the equation of the linear function.
- Determine the two parameters (a and b) of the equation by building up a system of two equations according to the stated problem.
- A point is on the graph of a function if and only if its coordinates fulfil the equation of the function.

R1.8 a) 12 supplied, 14 demanded

b) $f_s(q) = f_d(q)$ for $q = \frac{38}{3} = 12.6... \notin \mathbb{N} \Rightarrow$ no exact equilibrium $\rightarrow$ $q = 13$, $f_s(13) = 57$, $f_d(13) = 55$

R1.9 a) 4500 CHF

b) 388 CHF

c) C(x) = R(x) for x = 200

R1.10 Supply function $f_s(q) = \frac{1}{2}q$
Demand function $f_d(q) = -\frac{1}{2}q + 300$

Market equilibrium: $f_s(q) = f_d(q)$ for q = 300 and p = \$150

R1.11 a) S = {0, 4/3}

Hints:
- Factorise the left hand side of the equation (factor x).
- A product is equal to 0 if and only if at least one factor is 0.

b) S = {-1, 3}

Hint:
- Use the quadratic formula

c) S = { }

Hints:
- First solve for x^2.
- The square of any real number is equal to or greater than 0.

d) $S = \{2/3\}$

Hints:
- First get rid of the fractions by multiplying by the least common denominator (= 3x).
- The fractions $\frac{1}{x}$ and $\frac{x+1}{x}$ are not defined for x = 0. Hence, x = 0 cannot be a solution of the equation.

e) $S = \{-3\}$

Hints:
- First get rid of the fractions by multiplying by the least common denominator (= x(x-5)).
- The fractions in the original equation are not defined for x = 5. Hence, x = 5 cannot be a solution.

R1.12 a) $y = f(x) = -(x-2)^2 + 4$

b) $y = f(x) = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$

Hints:
- First state the equation of the quadratic function.
- Use the vertex form of the quadratic function in a).
- Use the general form of the quadratic function in b).
- Determine the parameters in the equation by building up a system of equations according to the stated problem.
- A point is on the graph of a function if and only if its coordinates fulfil the formula of the function.

R1.13 Supply function $f_s(q) = q^2 + 300$
Demand function $f_d(q) = -q + 410$

Market equilibrium: $f_s(q) = f_d(q)$ for q = 10 and p = 400

R1.14 C(x) = R(x)
$x_1 = 46 + 2\sqrt{89}$, $x_2 = 46 - 2\sqrt{89}$

R1.15 $k_1 = 0, k_2 = -1$

Hints:
- The position of the graph of f depends on the value of the parameter k.
- k must be such that the equation f(x) = 0 has exactly one solution.
- Expand the expression $(k-x)(x-2) - k(x^2-2) - 1$
- f is a linear function if k = -1 (quadratic terms cancel out). If $k \neq -1$, f is a quadratic function.
- In case f is quadratic, the equation f(x) = 0 has exactly one solution if the discriminant (term under square root in quadratic formula) is 0. This condition leads to an equation in k that can be solved for k.
- In case f is linear (if k = -1), the equation f(x) = 0 has exactly one solution.

R1.16 $a_1 = 2, a_2 = 6$

Hints:
- The equation f(x) = g(x) must have exactly one solution.
- The equation f(x) = g(x) is a quadratic equation. It has exactly one solution if the discriminant (term under square root in quadratic formula) is 0. This condition leads to an equation in a that can be solved for a.

R1.17 a) $y = f(x) = 3^x$

b) $y = f(x) = 4 \cdot 5^x$

Hints:
- First state the equation of the exponential function.
- Determine the parameters in the equation by building up a system of equations according to the stated problem.
- A point is on the graph of a function if and only if its coordinates fulfil the equation of the function.

R1.18 a) 0

b) 3

c) - 1

d) 8

Hint:
- The expression $\log_a(x)$ is the answer to the question "a to what power is equal to x?"

e) 5

f) 3.15

Hint:
- Use the identity $a^{\log_a(x)} = x$ for $a \in \mathbb{R}^+ \backslash \{1\}$.

R1.19 a) $x = 1.5$

b) $x = 51.22...$

c) $x = 40.23...$

Hints:
- Isolate a term of the form a^{bx}.
- Calculate the logarithm of the term.
- Use the identity $\log_a(u^v) = v \cdot \log_a(u)$

R1.20 Simple interest

$C_n = C_0(1 + nr)$ where $C_0 = \$8000$, $r = 12\%$, $n = 3$

$\Rightarrow C_3 = \$10'880$

R1.21 Simple interest

$r = \frac{\frac{C_n}{C_0} - 1}{n}$ where $C_0 = \$2000$, $C_n = \$2100$, $r = 12\%$, $n = \frac{3}{4}$ (9 months = $\frac{3}{4}$ years)

$\Rightarrow r = 6\frac{2}{3}\%$

R1.22 Simple interest

$C_0 = \frac{C_n}{1 + nr}$ where $C_n = \$3000$, $r = 6\%$, $n = \frac{1}{3}$

$\Rightarrow C_0 = \$2941.18$

R1.23 Compound interest

$C_n = C_0 \left(1 + \frac{r}{m}\right)^{mn}$ where C_0 = \$1000, r = 8%, m = 4, n = 4

$\Rightarrow$ $C_n - C_0$ = \$372.79

R1.24 Compound interest

$C_0 = \frac{C_n}{\left(1 + \frac{r}{m}\right)^{mn}}$ where C_n = \$18'000, r = 5.4%, m = 12, n = 4

$\Rightarrow$ C_0 = \$14'510.26

R1.25 9.86 million (rounded)

Hints:
- The population grows exponentially.
- State the general form of the exponential function.
- Find out both the initial value and the growth factor.

Detailed answer:
- $y = f(x) = c \cdot a^x$
- initial value (population in 1990): c = f(0) = 4'500'000
- growth factor a = 1 + 4% = 1.04
- population in 2010: $f(20) = 4'500'000 \cdot 1.04^{20}$ = 9.86 Mio (rounded)

R1.26 Annuity due

$p = \frac{A_n(q-1)}{q(q^n-1)}$ where A_n = \$250'000, $q = 1 + \frac{10.2\%}{4}$, n = 18 (4 1/2 years = 18 quarters)

$\Rightarrow$ p = \$10'841.24

R1.27 Ordinary annuity

$n = \frac{\log_a\left(\frac{p}{p - A_0(q-1)}\right)}{\log_a(q)}$ where A_0 = \$488'000, p = \$40'000, $q = 1 + \frac{6.8\%}{2}$, a := 10

$\Rightarrow$ n = 16.02... $\rightarrow$ 16 half-years = 8 years

R1.28 2 annuities: 3 years starting from now (paying in money), 4 months (withdrawing money)

- 4 months (withdrawing money): annuity due

$A_0 = p \frac{q^n - 1}{q^{n-1}(q-1)}$ where p = \$5000, $q = 1 + \frac{6.6\%}{12}$, n = 4

$\Rightarrow$ A_0 = \$19'836.49…

- 3 years starting from now (paying in money): annuity due

$p = \frac{A_n(q-1)}{q(q^n-1)}$ where A_n = … (= A_0 in first annuity), $q = 1 + \frac{6.6\%}{12}$, n = 36

$\Rightarrow$ p = \$497.04 (rounded)

R1.29 The whole sum Mr. S pays in at the end of the first year pays interest. The capital at the end of the 8th year must be the same as the value the annuity would have if Mr. S made 8 payments at the end of each year.

- Ordinary annuity

$A_n = p\,\frac{q^n - 1}{q - 1}$ where p = 25'000 CHF, q = 1.045, n = 8

⇒ A_n = 234'500.34… CHF

- Compound interest

$C_0 = \frac{C_n}{q^n}$ where C_n = … (= A_n in annuity), q = 1.045, n = 7

⇒ C_n = 172'317.53 CHF (rounded up)

R1.30 a) Ordinary annuity (from age 60 to age 75)

$A_0 = p\,\frac{q^n - 1}{q^n(q - 1)}$ where p = 8000 CHF, q = 1.025, n = 15

⇒ A_0 = 99'051.02… CHF

- Compound interest (from age 55 to age 60)

$C_0 = \frac{C_n}{q^n}$ where C_n = … (= A_0 in annuity from age 60 to age 75), q = 1.03, n = 5

⇒ C_0 = 85'442.28… CHF

- Ordinary annuity (from age 25 to age 55)

$p = \frac{A_n(q - 1)}{q^n - 1}$ where A_n = … (= C_0), q = 1.03, n = 30

⇒ p = 1795.93 CHF

b) Compound interest (from age 25 to age 60)

$C_n = C_0\, q^n$ where C_0 = 40'000 CHF, q = 1.03, n = 35

⇒ C_n = 112'554.50 (rounded) > … (= A_0 in annuity from age 60 to age 75)

⇒ The amount is sufficient for his retirement scheme.