Exercises 16 Indefinite integral
Antiderivative, indefinite integral, coefficient/sum rule

Objectives

- be able to determine an antiderivative and the indefinite integral of a constant/basic power/basic exponential function.
- be able to apply the coefficient/sum rule to determine the indefinite integral of a function.
- be able to determine the cost/average cost/revenue/profit function if the marginal cost/average cost/revenue/profit function is known.

Problems

16.1 Determine the indefinite integrals below:

a) $\int x^3 \, dx$

b) $\int x^2 \, dx$

c) $\int \frac{1}{x^4} \, dx$

d) $\int \frac{1}{x^2} \, dx$

e) $\int x^{-5} \, dx$

f) $\int 4 \, dx$

g) $\int (-7) \, dx$

h) $\int e^x \, dx$

16.2 Determine the indefinite integral of the following functions f:

a) $f(x) = x^5$

b) $f(x) = 3x^2$

c) $f(x) = x^3 + 2x^2 - 5$

d) $f(x) = \frac{1}{2}x^5 - \frac{2}{3x^2}$

e) $f(x) = \frac{1}{2}x^3 - 2x^2 + 4x - 5$

f) $f(x) = x^{10} - \frac{1}{2}x^3 - x$

16.3 Find two antiderivatives $F_1(x)$ and $F_2(x)$ of $f(x)$ such that the stated conditions are fulfilled.

a) $f(x) = 10x^2 + x$ $F_1(0) = 3$ $F_2(0) = -1$

b) $f(x) = x^3 + 3x + 1$ $F_1(2) = 5$ $F_2(4) = -8$

16.4 Suppose that we know the equation of the derivative f' of a function f:

$$f'(x) = 3x^2 - 50x + 250$$

Determine the equation of the function f, if …

a) … $f(0) = 500$.

b) … $f(10) = 2500$.

16.5 Suppose that we know the equation of the second derivative f'' of a function f:

$$f''(x) = 2x - 1$$

Find the equation of …

a) … the first derivative f' such that $f'(2) = 4$.

b) … the function f such that $f'(2) = 4$ and $f(1) = -1$.

16.6 If the monthly marginal cost (in dollars) for a product is $C'(x) = 2x + 100$, with fixed costs amounting to $200, find the total cost function for the month.

16.7 If the marginal cost (in dollars) for a product is $C'(x) = 4x + 2$, and the production of 10 units results in a total cost of $300, find the total cost function.

16.8 If the marginal cost (in dollars) for a product is $C'(x) = 4x + 40$, and the total cost of producing 25 units is $3000, what will be the cost of producing 30 units?

16.9 A firm knows that its marginal cost for a product is $C'(x) = 3x + 20$, that its marginal revenue is $R'(x) = 44 - 5x$, and that the cost of production and sale of 80 units is $11'400.

a) Find the profit function P(x).

b) How many units will result in a maximum profit?

Hint:
- The revenue R is zero if no unit is sold. Thus, R(0) = $0.

16.10 Suppose that the marginal revenue R'(x) and the derivative of the average cost $\overline{C}'(x)$ are given as follows:

$$R'(x) = 100$$
$$\overline{C}'(x) = 2 - \frac{1800}{x^2}$$

The production of 10 units results in a total cost of $1000.

a) Find the total cost function C(x).

b) How many units will result in a maximum profit? Find the maximum profit.

Answers

16.1 a) $\int x^3\,dx = \frac{x^4}{4} + C$

b) $\int x^2\,dx = \frac{x^3}{3} + C$

c) $\int \frac{1}{x^4}\,dx = -\frac{1}{3x^3} + C$

d) $\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$

e) $\int x^{-5}\,dx = -\frac{1}{4x^4} + C$

f) $\int 4\,dx = 4x + C$

g) $\int (-7)\,dx = -7x + C$

h) $\int e^x\,dx = e^x + C$

16.2 a) $\int f(x)\,dx = \int x^5\,dx = \frac{x^6}{6} + C$

b) $\int f(x)\,dx = \int 3x^2\,dx = x^3 + C$

c) $\int f(x)\,dx = \int \left(x^3 + 2x^2 - 5\right)dx = \frac{x^4}{4} + \frac{2x^3}{3} - 5x + C$

d) $\int f(x)\,dx = \int \left(\frac{1}{2}x^5 - \frac{2}{3x^2}\right)dx = \frac{x^6}{12} + \frac{2}{3x} + C$

e) $\int f(x)\,dx = \int \left(\frac{1}{2}x^3 - 2x^2 + 4x - 5\right)dx = \frac{x^4}{8} - \frac{2x^3}{3} + 2x^2 - 5x + C$

f) $\int f(x)\,dx = \int \left(x^{10} - \frac{1}{2}x^3 - x\right)dx = \frac{x^{11}}{11} - \frac{x^4}{8} - \frac{x^2}{2} + C$

16.3 a) $F_1(x) = \frac{10x^3}{3} + \frac{x^2}{2} + 3$ $F_2(x) = \frac{10x^3}{3} + \frac{x^2}{2} - 1$

b) $F_1(x) = \frac{x^4}{4} + \frac{3x^2}{2} + x - 7$ $F_2(x) = \frac{x^4}{4} + \frac{3x^2}{2} + x - 100$

Hints:
- First, determine the indefinite integral of f(x).
- Then, find the value of the integration constant such that the stated condition is fulfilled.

16.4 a) $f(x) = x^3 - 25x^2 + 250x + 500$

b) $f(x) = x^3 - 25x^2 + 250x + 1500$

16.5 a) $f'(x) = x^2 - x + 2$

b) $f(x) = \frac{x^3}{3} - \frac{x^2}{2} + 2x - \frac{17}{6}$

16.6 $C(x) = x^2 + 100x + 200$

Hints:
- First integrate the marginal cost function C'(x) $\Rightarrow$ $C(x) = x^2 + 100x + C$ ($C \in \mathbb{R}$)
- Determine the integration constant C using the fact that C(0) = \$200 $\Rightarrow$ C = 200

16.7 $C(x) = 2x^2 + 2x + 80$

16.8 $C(30) = \$3750$

Hint:
- First, determine the cost function $C(x) \Rightarrow C(x) = 2x^2 + 40x + 750$.

16.9 a) $P(x) = -4x^2 + 24x - 200$

Hints:
- Find the cost and revenue functions C(x) and R(x) $\Rightarrow C(x) = \frac{3}{2}x^2 + 20x + 200$, $R(x) = 44x - \frac{5}{2}x^2$
- Then, determine the profit function P(x).

b) $x = 3$

Hints:
- Find the relative maximum of the profit function P(x).
- Check if the relative maximum is the absolute maximum.

16.10 a) $C(x) = 2x^2 - 100x + 1800$

Hints:
- First, determine the average cost function $\overline{C}(x) \Rightarrow \overline{C}(x) = 2x + \frac{1800}{x} + C_1$
- Then, determine the cost function C(x).

b) P = \$3200 is the absolute maximum profit at x = 50 units.

Hints:
- First, determine the revenue function $R(x) \Rightarrow R(x) = 100x$
- Then, find the profit function $P(x) \Rightarrow P(x) = -2x^2 + 200x - 1800$
- Find the relative maximum of the profit function P(x).
- Check if the relative maximum is the absolute maximum.