

Review exercises 2 Differential calculus, integral calculus

Problems

R2.1 Decide whether the following statements are true or false:

- a) "The derivative of a function is a function."
- b) "The derivative of a function at a particular value of the variable is a number."
- c) "The function f has a relative maximum at $x = x_1$ if $f'(x_1) = 0$ and $f''(x_1) > 0$."
- d) "If $f'(x_2) = 0$ and $f''(x_2) < 0$, then the function f has a point of inflection at $x = x_2$."
- e) "Suppose that the function f has a relative maximum at $x = x_1$. If there are no other relative maxima and if there is no relative minimum at all, then the relative maximum is the absolute maximum of f ."
- f) "If $g' = f$, then g is an antiderivative of f ."
- g) "f with $f(x) = 2x + 20$ is an antiderivative of g with $g(x) = x^2$."
- h) "f with $f(x) = 3x$ has infinitely many antiderivatives."
- i) "The indefinite integral of a function is a set of functions."

R2.2 Determine the value $f(x_0)$, the first derivative $f'(x_0)$, and the second derivative $f''(x_0)$ at x_0 for the following functions f :

- a) $f(x) = 4x^2(x^2 - 1)$
 - i) $x_0 = 0$ ii) $x_0 = -1$
- b) $f(x) = (-3x^2 + 2x - 1) \cdot e^x$
 - i) $x_0 = 0$ ii) $x_0 = -2$
- c) $f(x) = (x^2 + 2) \cdot e^{-3x}$
 - i) $x_0 = 1$ ii) $x_0 = -\frac{1}{3}$

R2.3 For the given cost function $C(x)$ and revenue function $R(x)$ determine ...

- i) ... the marginal cost function $C'(x)$.
- ii) ... the marginal revenue function $R'(x)$.
- iii) ... the marginal profit function $P'(x)$.
- a) $C(x) = 200 + 40x$ $R(x) = 60x$
- b) $C(x) = 100 + 20x + 5x^2$ $R(x) = 100x - 2x^2$
- c) $C(x) = 50 + 20x^2 + 3e^{4x}$ $R(x) = 200x - e^{-4x^2}$

R2.4 For each function, find ...

- i) ... the relative maxima and minima.
- ii) ... the points of inflection.
- a) $f(x) = 2x^3 - 9x^2 + 12x - 1$
- b) $f(x)$ as in R2.2 a)

R2.5 The total revenue function for a commodity is given by

$$R(x) = 36x - 0.01x^2$$

Find the maximum revenue ...

- a) ... if production is not limited to a certain number of units.
- b) ... if production is limited to at most 1500 units.

R2.6 If the total cost function for a product is

$$C(x) = 100 + x^2$$

producing how many units x will result in a minimum average cost per unit? Find the minimum average cost.

R2.7 A firm can produce only 1000 units per month. The monthly total cost is given by

$$C(x) = 300 + 200x$$

dollars, where x is the number produced. If the total revenue is given by

$$R(x) = 250x - \frac{1}{100}x^2$$

dollars, how many items should the firm produce for maximum profit? Find the maximum profit.

R2.8 Determine the indefinite integrals below:

a) $\int (x^4 - 3x^3 - 6) dx$

b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4} \right) dx$

R2.9 The equation of the third derivative f''' of a function f is given as follows:

$$f'''(x) = 3x + 1$$

Find the equation of the function f such that $f''(0) = 0$, $f'(0) = 1$, $f(0) = 2$

R2.10 If the marginal cost (in dollars) for producing a product is $C'(x) = 5x + 10$, with a fixed cost of \$800, what will be the cost of producing 20 units?

R2.11 A certain firm's marginal cost $C'(x)$ and the derivative of the average revenue $\bar{R}'(x)$ are given as follows:

$$C'(x) = 6x + 60$$

$$\bar{R}'(x) = -1$$

The total cost and revenue of the production of 10 items are \$1000 and \$1700, respectively.

How many units will result in a maximum profit? Find the maximum profit.

R2.12 The demand function for a product is

$$p = f(x) = 49 - x^2$$

and the supply function is

$$p = g(x) = 4x + 4$$

Find the equilibrium point and both the consumer's and the producer's surplus there.

R2.13 The demand function for a product is

$$p = f(x) = 110 - ax^2$$

and the supply function is

$$p = g(x) = 2 - \frac{6}{5}x + bx^2$$

with unknown parameters a and b . The equilibrium price is \$10, and the producer's surplus is \$73.33

Determine the two unknown parameters a and b .

Answers

- R2.1 a) true b) true c) false
 d) true e) true f) true
 g) false h) true i) true

- R2.2 a) $f(x) = 16x^3 - 8x$
 $f'(x) = 48x^2 - 8$
 i) $f(0) = 0$ $f'(0) = 0$ $f''(0) = -8$
 ii) $f(-1) = 0$ $f'(-1) = -8$ $f''(-1) = 40$
- b) $f(x) = (-3x^2 - 4x + 1) \cdot e^x$
 $f'(x) = (-3x^2 - 10x - 3) \cdot e^x$
 i) $f(0) = -1$ $f'(0) = 1$ $f''(0) = -3$
 ii) $f(-2) = -17 \cdot e^{-2} = -2.300\dots$
 $f'(-2) = -3 \cdot e^{-2} = -0.406\dots$
 $f''(-2) = 5 \cdot e^{-2} = 0.676\dots$
- c) $f(x) = (-3x^2 + 2x - 6) \cdot e^{-3x}$
 $f'(x) = (9x^2 - 12x + 20) \cdot e^{-3x}$
 i) $f(1) = 3 \cdot e^{-3} = 0.149\dots$
 $f'(1) = -7 \cdot e^{-3} = -0.348\dots$
 $f''(1) = 17 \cdot e^{-3} = 0.846\dots$
 ii) $f\left(-\frac{1}{3}\right) = \frac{19}{9}e = 5.738\dots$
 $f'\left(-\frac{1}{3}\right) = -7e = -19.027\dots$
 $f''\left(-\frac{1}{3}\right) = 25e = 67.957\dots$

- R2.3 a) i) $C'(x) = 40$ ii) $R'(x) = 60$
 iii) $P'(x) = 20$
- b) i) $C'(x) = 20 + 10x$ ii) $R'(x) = 100 - 4x$
 iii) $P'(x) = 80 - 14x$
- c) i) $C'(x) = 40x + 12e^{4x}$ ii) $R'(x) = 200 + 8x e^{-4x^2}$
 iii) $P'(x) = 200 - 40x - 12e^{4x} + 8x e^{-4x^2}$

- R2.4 a) $f(x) = 2x^3 - 9x^2 + 12x - 1$
 $f'(x) = 6x^2 - 18x + 12$
 $f''(x) = 12x - 18$
 $f'''(x) = 12$
 i) $f'(x) = 0$ at $x_1 = 1$ and $x_2 = 2$
 $f''(x_1) = -6 < 0$ \Rightarrow relative maximum at $x_1 = 1$
 $f''(x_2) = 6 > 0$ \Rightarrow relative minimum at $x_2 = 2$

ii) $f'(x) = 0$ at $x_3 = \frac{3}{2}$
 $f''(x_3) = 12 \neq 0 \Rightarrow$ point of inflection at $x_3 = \frac{3}{2}$

b) $f(x) = 4x^2(x^2 - 1)$
 $f'(x) = 16x^3 - 8x = 8x(2x^2 - 1)$
 $f''(x) = 48x^2 - 8 = 8(6x^2 - 1)$
 $f'''(x) = 96x$

i) $f'(x) = 0$ at $x_1 = 0$, $x_2 = \frac{1}{\sqrt{2}}$, and $x_3 = -\frac{1}{\sqrt{2}}$
 $f''(x_1) = -8 < 0 \Rightarrow$ relative maximum at $x_1 = 0$
 $f''(x_2) = 16 > 0 \Rightarrow$ relative minimum at $x_2 = \frac{1}{\sqrt{2}}$
 $f''(x_3) = 16 > 0 \Rightarrow$ relative minimum at $x_3 = -\frac{1}{\sqrt{2}}$

ii) $f'(x) = 0$ at $x_3 = \frac{1}{\sqrt{6}}$
 $f''(x_3) = \frac{96}{\sqrt{6}} \neq 0 \Rightarrow$ point of inflection at $x_3 = \frac{1}{\sqrt{6}}$

R2.5 a) **Relative** maximum at $x_1 = 1800$
 $R(x_1) = \$32'400$
 $R(x) < R(x_1)$ if $x \neq x_1$ as there is no relative minimum
 $\Rightarrow R = \$32'400$ is the **absolute** maximum revenue at $x = 1800$.

b) Relative maximum at $x = 1800$ lies outside the possible interval $0 \leq x \leq 1500$
 $R(1500) = \$31'500 > R(0) = 0\$$
 $\Rightarrow R = \$31'500$ is the **absolute** maximum revenue at $x = 1500$.

R2.6 $\bar{C}(x) = \frac{C(x)}{x} = \frac{100}{x} + x$
 $\bar{C}(x)$ has a **relative** minimum at $x_1 = 10$
 $\bar{C}(20) = \$20$
 $\bar{C}(x) > \bar{C}(x_1)$ if $x \neq x_1$ as there is no relative maximum
 $\Rightarrow \bar{C} = \$20$ is the **absolute** minimum average cost at $x = 10$.

R2.7 $P(x) = R(x) - C(x) = -\frac{1}{100}x^2 + 50x - 300$
 $P(x)$ has a **relative** maximum at $x_1 = 2500$. This is outside the possible interval $0 \leq x \leq 1000$
 $P(1000) = \$39'700 > P(0) = -300\$$
 $\Rightarrow P = \$39'700$ is the **absolute** maximum profit at the endpoint $x = 1000$.

R2.8 a) $\int(x^4 - 3x^3 - 6) dx = \frac{x^5}{5} + \frac{3x^4}{4} - 6x + C$

b) $\int\left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right) dx = \frac{x^7}{14} - \frac{2}{9x^3} + C$

R2.9 $f(x) = \frac{x^4}{8} + \frac{x^3}{6} + x + 2$

R2.10 $C(20) = \$2000$

Hint:

- First, determine the cost function $C(x) \Rightarrow C(x) = \frac{5}{2}x^2 + 10x + 800$

R2.11 $P = \$800$ is the absolute maximum profit at $x = 15$ units.

Hints:

- Determine the cost function $C(x) \Rightarrow C(x) = 3x^2 + 60x + 100$
- Determine the average revenue function $\bar{R}(x) \Rightarrow \bar{R}(x) = -x + C$
- Determine the revenue function $R(x) \Rightarrow R(x) = -x^2 + 180x$
- Find the profit function $P(x) \Rightarrow P(x) = -4x^2 + 120x - 100$
- Find the relative maximum of the profit function $P(x)$.
- Check if the relative maximum is the absolute maximum.

R2.12 Equilibrium quantity $x = 5$
Equilibrium price $p = 24$
Consumer's surplus $CS = 83.33$
Producer's surplus $PS = 50$

R2.13 $a = 1$
 $b = 0.2$