Review exercises 2 Differential calculus, integral calculus

Problems

R2.1 Decide whether the following statements are true or false:

a) "The derivative of a function is a function."

b) "The derivative of a function at a particular value of the variable is a number."

c) "The function f has a relative maximum at $x = x_1$ if $f'(x_1) = 0$ and $f''(x_1) > 0$."

d) "If $f''(x_2) = 0$ and $f'''(x_2) < 0$, then the function f has a point of inflection at $x = x_2$."

e) "Suppose that the function f has a relative maximum at $x = x_1$. If there are no other relative maxima and if there is no relative minimum at all, then the relative maximum is the absolute maximum of f."

f) "If g' = f , then g is an antiderivative of f."

g) "f with $f(x) = 2x + 20$ is an antiderivative of g with $g(x) = x^2$."

h) "f with $f(x) = 3x$ has infinitely many antiderivatives."

i) "The indefinite integral of a function is a set of functions."

R2.2 Determine the value $f(x_0)$, the first derivative $f'(x_0)$, and the second derivative $f''(x_0)$ at x_0 for the following functions f:

a) $f(x) = 4x^2(x^2 - 1)$

i) $x_0 = 0$

ii) $x_0 = -1$

b) $f(x) = (- 3x^2 + 2x - 1)\cdot e^x$

i) $x_0 = 0$

ii) $x_0 = -2$

c) $f(x) = (x^2 + 2)\cdot e^{-3x}$

i) $x_0 = 1$

ii) $x_0 = -\frac{1}{3}$

R2.3 For each function, find …

i) … the relative maxima and minima.

ii) … the points of inflection.

a) $f(x) = 2x^3 - 9x^2 + 12x - 1$

b) f(x) as in R2.2 a)

R2.4 The total revenue function for a commodity is given by

$$R(x) = 36x - 0.01x^2$$

Find the maximum revenue …

a) ... if production is not limited to a certain number of units.

b) … if production is limited to at most 1500 units.

R2.5 If the total cost function for a product is

$$C(x) = 100 + x^2$$

producing how many units x will result in a minimum average cost per unit? Find the minimum average cost.

R2.6 A firm can produce only 1000 units per month. The monthly total cost ist given by

$$C(x) = 300 + 200x$$

dollars, where x is the number produced. If the total revenue is given by

$$R(x) = 250x - \frac{1}{100}x^2$$

dollars, how many items should the firm produce for maximum profit? Find the maximum profit.

R2.7 Determine the indefinite integrals below:

a) $\int \left(x^4 - 3x^3 - 6\right)\,dx$

b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right)\,dx$

R2.8 The equation of the third derivative f''' of a function f is given as follows:

$$f'''(x) = 3x + 1$$

Find the equation of the function f such that $f''(0) = 0$, $f'(0) = 1$, $f(0) = 2$

R2.9 If the marginal cost (in dollars) for producing a product is $C'(x) = 5x + 10$, with a fixed cost of \$800, what will be the cost of producing 20 units?

R2.10 A certain firm's marginal cost $C'(x)$ and the derivative of the average revenue $\overline{R}'(x)$ are given as follows:

$$C'(x) = 6x + 60$$
$$\overline{R}'(x) = -1$$

The total cost and revenue of production of 10 items are \$1000 and \$1700, respectively.

How many units will result in a maximum profit? Find the maximum profit.

Answers

R2.1	a)	true	b)	true	c)	false
	d)	true	e)	true	f)	true
	g)	false	h)	true	i)	true

R2.2 a)

$f'(x) = 16x^3 - 8x$
$f''(x) = 48x^2 - 8$

i) $f(0) = 0$ $f'(0) = 0$ $f''(0) = -8$

ii) $f(-1) = 0$ $f'(-1) = -8$ $f''(-1) = 40$

b)

$f'(x) = (-3x^2 - 4x + 1)\cdot e^x$
$f''(x) = (-3x^2 - 10x - 3)\cdot e^x$

i) $f(0) = -1$ $f'(0) = 1$ $f''(0) = -3$

ii) $f(-2) = -17\cdot e^{-2} = -2.300\ldots$
$f'(-2) = -3\cdot e^{-2} = -0.406\ldots$
$f''(-2) = 5\cdot e^{-2} = 0.676\ldots$

c)

$f'(x) = (-3x^2 + 2x - 6)\cdot e^{-3x}$
$f''(x) = (9x^2 - 12x + 20)\cdot e^{-3x}$

i) $f(1) = 3\cdot e^{-3} = 0.149\ldots$
$f'(1) = -7\cdot e^{-3} = -0.348\ldots$
$f''(1) = 17\cdot e^{-3} = 0.846\ldots$

ii) $f\left(-\frac{1}{3}\right) = \frac{19}{9}e = 5.738\ldots$
$f'\left(-\frac{1}{3}\right) = -7e = -19.027\ldots$
$f''\left(-\frac{1}{3}\right) = 25e = 67.957\ldots$

R2.3 a)

$f(x) = 2x^3 - 9x^2 + 12x - 1$
$f'(x) = 6x^2 - 18x + 12$
$f''(x) = 12x - 18$
$f'''(x) = 12$

i) $f'(x) = 0$ at $x_1 = 1$ and $x_2 = 2$
$f''(x_1) = -6 < 0$ $\Rightarrow$ relative maximum at $x_1 = 1$
$f''(x_2) = 6 > 0$ $\Rightarrow$ relative minimum at $x_2 = 2$

ii) $f''(x) = 0$ at $x_3 = \frac{3}{2}$
$f'''(x_3) = 12 \neq 0$ $\Rightarrow$ point of inflection at $x_3 = \frac{3}{2}$

b)

$f(x) = 4x^2(x^2 - 1)$
$f'(x) = 16x^3 - 8x = 8x(2x^2 - 1)$
$f''(x) = 48x^2 - 8 = 8(6x^2 - 1)$
$f'''(x) = 96x$

i) $f'(x) = 0$ at $x_1 = 0$, $x_2 = \frac{1}{\sqrt{2}}$, and $x_3 = -\frac{1}{\sqrt{2}}$
$f''(x_1) = -8 < 0$ $\Rightarrow$ relative maximum at $x_1 = 0$
$f''(x_2) = 16 > 0$ $\Rightarrow$ relative minimum at $x_2 = \frac{1}{\sqrt{2}}$
$f''(x_3) = 16 > 0$ $\Rightarrow$ relative minimum at $x_3 = -\frac{1}{\sqrt{2}}$

ii) $f''(x) = 0$ at $x_3 = \frac{1}{\sqrt{6}}$
$f'''(x_3) = \frac{96}{\sqrt{6}} \neq 0$ $\Rightarrow$ point of inflection at $x_3 = \frac{1}{\sqrt{6}}$

R2.4 a) **Relative** maximum at $x_1 = 1800$
$R(x_1) = \$32'400$
$R(x) < R(x_1)$ if $x \neq x_1$ as there is no relative minimum
$\Rightarrow$ R = \$32'400 is the **absolute** maximum revenue at x = 1800.

b) Relative maximum at x = 1800 lies outside the possible interval $0 \le x \le 1500$
R(1500) = \$31'500 > R(0) = 0\$
$\Rightarrow$ R = \$31'500 is the **absolute** maximum revenue at x = 1500.

R2.5 $\overline{C}(x) = \frac{C(x)}{x} = \frac{100}{x} + x$
$\overline{C}(x)$ has a **relative** minimum at $x_1 = 10$
$\overline{C}(20) = \$20$
$\overline{C}(x) > \overline{C}(x_1)$ if $x \neq x_1$ as there is no relative maximum
$\Rightarrow$ $\overline{C}$ = \$20 is the **absolute** minimum average cost at x = 10.

R2.6 $P(x) = R(x) - C(x) = -\frac{1}{100}x^2 + 50x - 300$
P(x) has a **relative** maximum at $x_1 = 2500$. This is outside the possible interval $0 \le x \le 1000$
P(1000) = \$39'700 > P(0) = - 300\$
$\Rightarrow$ P = \$39'700 is the **absolute** maximum profit at the endpoint x = 1000.

R2.7 a) $\int \left(x^4 - 3x^3 - 6\right)\, dx = \frac{x^5}{5} + \frac{3x^4}{4} - 6x + C$

b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right)\, dx = \frac{x^7}{14} - \frac{2}{9x^3} + C$

R2.8 $f(x) = \frac{x^4}{8} + \frac{x^3}{6} + x + 2$

R2.9 C(20) = \$2000

Hint:
- First, determine the cost function C(x) $\Rightarrow$ $C(x) = \frac{5}{2}x^2 + 10x + 800$

R2.10 P = \$800 is the absolute maximum profit at x = 15 units.

Hints:
- Determine the cost function C(x) $\Rightarrow$ $C(x) = 3x^2 + 60x + 100$
- Determine the average revenue function $\overline{R}(x)$ $\Rightarrow$ $\overline{R}(x) = - x + C$
- Determine the revenue function R(x) $\Rightarrow$ $R(x) = - x^2 + 180x$
- Find the profit function P(x) $\Rightarrow$ $P(x) = - 4x^2 + 120x - 100$
- Find the relative maximum of the profit function P(x).
- Check if the relative maximum is the absolute maximum.