Exercises 14 Differentiation rules
Coefficient/sum/product rule, chain rule, higher-order derivatives

Objectives

- be able to apply the coefficient, sum, product rule to determine the derivative of a function.
- be able to apply the chain rule to determine the derivative of a function.
- be able to determine a higher-order derivative of a function.

Problems

14.1 Determine the derivative by applying the **coefficient rule**:

a)	$f(x) = 3x^5$	b)	$f(x) = -4x^3$	c)	$f(x) = -x^{10}$
d)	$f(x) = a \cdot x^3$	e)	$f(x) = n \cdot x^{n-1}$	f)	$f(x) = 9 \cdot 3^x$
g)	$s(t) = \frac{1}{2} g \cdot t^2$	h)	$S(T) = \alpha \cdot T^4$	i)	$v(s) = \sqrt{2 \cdot s \cdot g}$

14.2 Determine the derivative by applying the **sum rule**:

a)	$f(x) = x^5 + x^6$	b)	$f(x) = x^{10} - x^9$	c)	$f(x) = x^{\frac{2}{3}} + x^{\frac{3}{2}}$
d)	$f(x) = 1 + x + 3x^3$	e)	$f(x) = \frac{1}{4} x^4 + 3x^2 - 2$	f)	$f(x) = -3x^8 + x^5 - 3x + 99$
g)	$f(x) = ax^2 + bx + c$	h)	$f(x) = 3(a^2 - 2ax + x^2)$	i)	$f(x) = \frac{x^3}{3} - \frac{3}{x^3}$
j)	$s(t) = s_0 + v_0 t + \frac{1}{2} g \cdot t^2$	k)	$V(r) = -\frac{a}{r} + \frac{b}{r^2}$	l)	$C(n) = C_0(1 + nr)$

14.3 Determine the derivative by applying the **product rule**:

a)	$f(x) = x \cdot e^x$	b)	$f(x) = x^3 \cdot 3^x$
c)	$f(x) = -2x^5(x - 1)$	d)	$f(x) = (2x - 1) \cdot e^x$
e)	$f(x) = -2x \cdot \sqrt{x}$	f)	$f(x) = (-3x^2 - x + 1) \cdot \sqrt[3]{x}$
g)	$f(x) = (2x - 1)(-3x^2 - x + 1)$	h)	$f(x) = 3(1 - x^2)(x^{10} - x^9)$
i)	$V(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3}\right)$	j)	$T(V) = \frac{1}{n \cdot R}\left(p + \frac{a \cdot n^2}{V^2}\right)(V - n \cdot b)$
k) *	$f(x) = x \cdot \sqrt{x} \cdot e^x$	l) *	$f(x) = (x^2 - 1) \cdot \sqrt[3]{x} \cdot e^x$

14.4 Determine the derivative by applying the **chain rule**:

a)	$f(x) = (2x)^3$	b)	$f(x) = (3x - 1)^5$	c)	$f(x) = (-3x^3 + x^2 - 4x)^6$
d)	$f(x) = e^{4x}$	e)	$f(x) = e^{-x}$	f)	$f(x) = e^{1 - \frac{x}{2}}$
g)	$f(x) = e^{-x^2}$	h)	$f(x) = e^{x^2 - 2x + 5}$	i)	$f(x) = e^{e^x}$
j)	$f(x) = 2^{3^x}$	k) *	$f(x) = 2^{e^{2x}}$	l) **	$f(x) = x^x$

14.5 Determine the derivative by applying the appropriate differentiation rule(s), and simplify the expression as far as possible:

a) $f(x) = (x - 2)\, e^{2x}$ b) $f(x) = (2 - x^2)e^{-x}$

c) $f(x) = (3x^3 - 2x^2 + x - 1)\, e^{-2x}$ d) $f(x) = \left(x - 2\right)^2 e^{-x^2 - 2x}$

e) $f(x) = ax\, e^{-\frac{x^2}{2}}$ f) $P(v) = av^2\, e^{-bv^2}$

14.6 Determine the derivative of the indicated function at the indicated value of the variable:

a) f in 14.1 b) $x = 2$ b) s in 14.1 g) $t = 4$

c) f in 14.2 g) $x = -1$ d) f in 14.5 e) $x = 0$

14.7 Determine the second and third derivatives of the functions in problem …

a) … 14.1 a) b) … 14.2 g)

c) … 14.3 a) d) … 14.4 g)

e) … 14.5 b) f) … 14.5 e)

14.8 Determine the indicated higher-order derivatives:

a) $f''(-1)$ with function f in 14.1 a)

Hint:
- You have already determined $f''(x)$ in 14.7 a).

b) $f'''(2)$ with function f in 14.5 e)

Hint:
- You have already determined $f'''(x)$ in 14.7 f).

Answers

14.1 a) $f'(x) = 3\cdot 5x^4 = 15x^4$

b) $f'(x) = (-4)\, 3x^2 = -12x^2$

c) $f'(x) = (-1)\, 10x^9 = -10x^9$

d) $f'(x) = a\cdot 3x^2 = 3ax^2$

Hint:
- a is a constant.

e) $f'(x) = n(n-1)x^{n-2}$

f) $f'(x) = 9\cdot 3^x\cdot \ln(3)$

g) $s'(t) = \frac{g}{2}\, 2t = gt$

Hints:
- The name of the function is s, and the variable is t.
- g is a constant.

h) $S'(T) = \alpha\cdot 4T^3 = 4\alpha T^3$

i) $v'(s) = \sqrt{2g}\,\frac{1}{2\sqrt{s}} = \sqrt{\frac{g}{2s}}$

Hint:
- v(s) can be rewritten as a product of two square roots.

14.2 a) $f'(x) = 5x^4 + 6x^5$ b) $f'(x) = 10x^9 - 9x^8$ c) $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{3}{2}x^{\frac{1}{2}}$

d) $f'(x) = 1 + 9x^2$ e) $f'(x) = x^3 + 6x$ f) $f'(x) = -24x^7 + 5x^4 - 3$

g) $f'(x) = 2ax + b$ h) $f'(x) = -6a + 6x$ i) $f'(x) = x^2 + \frac{9}{x^4}$

j) $s'(t) = v_0 + gt$ k) $V'(r) = \frac{a}{r^2} - \frac{b}{r^3}$ l) $C'(n) = C_0\cdot r$

14.3 a) $f'(x) = e^x + x\cdot e^x$

b) $f'(x) = 3x^2\cdot 3^x + x^3\cdot 3^x\cdot \ln(3)$

c) $f'(x) = -2\left(5x^4(x-1) + x^5\right)$

d) $f'(x) = 2\cdot e^x + (2x-1)\cdot e^x$

e) $f'(x) = -2\left(\sqrt{x} + x\cdot \frac{1}{2\sqrt{x}}\right)$

f) $f'(x) = (-6x-1)\cdot \sqrt[3]{x} + (-3x^2 - x + 1)\cdot \frac{1}{3\sqrt[3]{x^2}}$

g) $f'(x) = 2(-3x^2 - x + 1) + (2x-1)(-6x-1)$

h) $f'(x) = 3\left(-2x(x^{10} - x^9) + (1-x^2)(10x^9 + 9x^8)\right)$

i) $V'(r) = e^r\left(a\cdot r^2 - \frac{b}{r^3}\right) + e^r\left(2a\cdot r + \frac{3b}{r^4}\right)$

Hints:
- V is the name of the function, and r is the variable.
- a and b are constants.

j) $T'(V) = \frac{1}{n \cdot R}\left(-\frac{2a \cdot n^2}{V^3}\left(V - n \cdot b\right) + \left(p + \frac{a \cdot n^2}{V^2}\right)\right)$

Hints:
- T is the name of the function, and V is the variable.
- n, R, p, a and b are constants.

k) * $f'(x) = \left(\sqrt{x} \cdot e^x\right) + x\left(\sqrt{x} \cdot e^x\right)' = \left(\sqrt{x} \cdot e^x\right) + x\left(\frac{1}{2\sqrt{x}} \cdot e^x + \sqrt{x} \cdot e^x\right)$

Hints:
- f(x) is a product consisting of three factors.
- Think of f(x) as a product of two factors where the second factor itself consists of two factors, i.e. $f(x) = x \cdot \left(\sqrt{x} \cdot e^x\right)$
- The product rule has to be applied twice, once for the whole expression f(x) and once for the second factor in f(x).

l) * $f'(x) = 2x\left(\sqrt[3]{x} \cdot e^x\right) + \left(x^2 - 1\right)\left(\sqrt[3]{x} \cdot e^x\right)' = 2x\left(\sqrt[3]{x} \cdot e^x\right) + \left(x^2 - 1\right)\left(\frac{1}{3\sqrt[3]{x^2}} \cdot e^x + \sqrt[3]{x} \cdot e^x\right)$

Hint:
- Use the same procedure as in k).

14.4

a) $f(x) = 3(2x)^2 \cdot 2 = 24x^2$

b) $f(x) = 5(3x - 1)^4 \cdot 3 = 15(3x - 1)^4$

c) $f(x) = 6(-3x^3 + x^2 - 4x)^5 \cdot (-9x^2 + 2x - 4)$

d) $f(x) = e^{4x}\ 4 = 4\ e^{4x}$

e) $f(x) = e^{-x}\ (-1) = -e^{-x}$

f) $f(x) = e^{1 - \frac{x}{2}}\left(-\frac{1}{2}\right) = -\frac{1}{2}\,e^{1 - \frac{x}{2}}$

g) $f(x) = e^{-x^2}(-2x) = -2x \cdot e^{-x^2}$

h) $f(x) = e^{x^2 - 2x + 5}(2x - 2)$

i) $f(x) = e^{e^x} \cdot e^x$

j) $f(x) = 2^{3^x} \cdot \ln(2) \cdot 3^x \cdot \ln(3)$

k) * $f(x) = 2^{e^{2x}} \cdot \ln(2) \cdot e^{2x} \cdot 2$

l) ** $f(x) = x^x \cdot (\ln(x) + 1)$

Hints:
- The expression x^x can be rewritten as follows: $x^x = e^{\ln(x^x)} = e^{x \cdot \ln(x)}$
- The derivative of ln(x) is $\frac{1}{x}$

14.5

a) $f'(x) = e^{2x} + (x - 2)\ e^{2x}\ 2 = (2x - 3)\ e^{2x}$

b) $f'(x) = -2x\ e^{-x} + (2 - x^2)\ e^{-x}\ (-1) = (x^2 - 2x - 2)\ e^{-x}$

c) $f'(x) = (9x^2 - 4x + 1)\ e^{-2x} - 2(3x^3 - 2x^2 + x - 1)\ e^{-2x} = (-6x^3 + 13x^2 - 6x + 3)\ e^{-2x}$

d) $f'(x) = 2\left(x - 2\right)e^{-x^2-2x} + \left(x - 2\right)^2\left(-2x - 2\right)e^{-x^2-2x} = 2\ (x^3 + 3x^2 + x - 6)\ e^{-x^2-2x}$

e) $f'(x) = a\left(e^{-\frac{x^2}{2}} + x\ e^{-\frac{x^2}{2}}(-x)\right) = a\left(1 - x^2\right)e^{-\frac{x^2}{2}}$

f) $P'(v) = a\left(2v\ e^{-bv^2} + v^2 e^{-bv^2}\left(-2bv\right)\right) = 2av\left(1 - bv^2\right)e^{-bv^2}$

14.6

a) $f'(2) = -48$

b) $s'(4) = 4g$

c) $f'(-1) = -2a + b$

d) $f'(0) = a$

14.7 a) 14.1 a)
$f''(x) = 15 \cdot 4x^3 = 60x^3$
$f'''(x) = 60 \cdot 3x^2 = 180x^2$

b) 14.2 g)
$f''(x) = 2a \cdot 1 = 2a$
$f'''(x) = 0$

c) 14.3 a)
$f''(x) = e^x + (e^x + x \cdot e^x) = (x + 2)\, e^x$
$f'''(x) = e^x + (x + 2)\, e^x = (x + 3)\, e^x$

d) 14.4 g)
$f''(x) = -2\left(e^{-x^2} + x\, e^{-x^2}(-2x)\right) = 2\left(2x^2 - 1\right)e^{-x^2}$
$f'''(x) = 2\left(4x\, e^{-x^2} + \left(2x^2 - 1\right)e^{-x^2}(-2x)\right) = 4x\left(-2x^2 + 3\right)e^{-x^2}$

e) 14.5 b)
$f''(x) = (2x - 2)e^{-x} + (x^2 - 2x - 2)\, e^{-x}\, (-1) = (4x - x^2)\, e^{-x}$
$f'''(x) = (4 - 2x)\, e^{-x} + (4x - x^2)\, e^{-x}\, (-1) = (x^2 - 6x + 4)\, e^{-x}$

f) 14.5 e)
$f''(x) = a\left(-2x\, e^{-\frac{x^2}{2}} + (1 - x^2)\, e^{-\frac{x^2}{2}}(-x)\right) = a\left(x^3 - 3x\right) e^{-\frac{x^2}{2}}$
$f'''(x) = a\left(\left(3x^2 - 3\right) e^{-\frac{x^2}{2}} + \left(x^3 - 3x\right) e^{-\frac{x^2}{2}}(-x)\right) = a\left(-x^4 + 6x^2 - 3\right) e^{-\frac{x^2}{2}}$

14.8 a) $f''(-1) = -60$

b) $f'''(2) = a\left(-16 + 6 \cdot 4 - 3\right) e^{-\frac{4}{2}} = \frac{5a}{e^2}$