Equations

An **equation** consists of two terms which are connected with an **equality sign**.

Ex.: $7x - 3 = 12x - 38$
equation

$2(7x - 3) + 5x$
no equation

A **solution** of an equation in one **variable** is a number that, when substituted for the variable, satisfies the equation, i.e. forms a true statement.

A solution of an equation in two ore more variables is a set of numbers that, when substituted for the variables, satisfies the equation, i.e. forms a true statement.

Ex.: $2x + 4 = 10$
This equation has exactly one solution: $x = 3$

Ex.: $x = x + 1$
This equation has no solution.

Ex.: $y^2 - 1 = 3$
This equation has two solutions: $y_1 = 2$
$y_2 = -2$

Ex.: $\sqrt{x-1} + y = 2$
This equation has infinitely many solutions: $(x,y)_1 = (2,1)$
$(x,y)_2 = (5,0)$
$(x,y)_3 = (10,-1)$
etc.

The set of all the solutions of an equation is the **solution set** S.

Ex.: $2x + 4 = 10$ $S = \{ 3 \}$

Ex.: $x = x + 1$ $S = \{\}$

Ex.: $y^2 - 1 = 3$ $S = \{ 2 , -2 \}$

Ex.: $\sqrt{x-1} + y = 2$ $S = \{ (2,1) , (5,0) , (10,-1) , \dots \}$

Two equations with the same solution set are **equivalent**.

Ex.: $2x + 4 = 10$
$x + 1 = 4$
These two equations have the same solution set $S = \{ 3 \}$.
They are therefore equivalent.

Solving an equation

Solving an equation means rearranging the equation to the form "variable = ..." by applying one or more **equivalence operations**.

Ex.:
$2x + 4 = 10 \quad | - 4$
$2x = 6 \quad | : 2$
$x = 3$
$S = \{ 3 \}$

Ex.:
$(x+2)(x-3) - 3(2x-3) = (x-6)^2 + 2 \quad |$ expanding
$(x^2 - x - 6) - (6x - 9) = (x^2 - 12x + 36) + 2 \quad |$ dissolving brackets
$x^2 - x - 6 - 6x + 9 = x^2 - 12x + 36 + 2 \quad |$ simplifying
$x^2 - 7x + 3 = x^2 - 12x + 38 \quad | - x^2$
$- 7x + 3 = - 12x + 38 \quad | + 12x$
$5x + 3 = 38 \quad | - 3$
$5x = 35 \quad | : 5$
$x = 7$
$S = \{ 7 \}$

Equivalence operations

The following operations transform an equation into an equivalent equation. The new equation has therefore the same solution set as the original equation.

- **Addition** of an **arbitrary number** to both sides of the equation
- **Subtraction** of an **arbitrary number** from both sides of the equation
- **Multiplication** of both sides of the equation by an **arbitrary number ≠ 0**
- **Division** of both sides of the equation by an **arbitrary number ≠ 0**
- ...

Systems of equations

A **system of equations** consists of two or more equations.

Ex.: $2x + y = 5$
$x + 2y = 4$
System of 2 equations in 2 variables (x und y)

Ex.: $3p - 2r + 4s - t = 0$
$p^2 + q^2 = 1$
$p + q = r - s$
System of 3 equations in 5 variables (p, q, r, s, t)

A **solution of a system of equations** is a set of numbers that, when substituted for the variables, satisfies **each** equation.

Ex.: $2x + y = 5$ I
$x + 2y = 4$ II

Equation I has infinitely many solutions:
$(x,y)_1 = (0,5)$
$(x,y)_2 = (1,3)$
$(x,y)_3 =$ **(2,1)**
$(x,y)_4 = (3,-1)$
$(x,y)_5 = (4,-3)$
etc.

Equation II has infinitely many solutions, too:
$(x,y)_1 = (-2,3)$
$(x,y)_2 = (0,2)$
$(x,y)_3 =$ **(2,1)**
$(x,y)_4 = (4,0)$
$(x,y)_5 = (6,-1)$
etc.

Only the set $(x,y) = (2,1)$ satisfies both equation I and equation II.

Therefore, the system of equations has exactly one solution:
$(x,y) = (2,1)$

Solving a system of equations

1. Operations

- **Equivalence operation** applied to one single equation
  (see "Solving an equation" above)

  An equivalence operation does not change the solution set of a single equation.

  Ex.: $2x + y = 5 \quad | \cdot 2$
  $4x + 2y = 10$

  Both equations have the same solutions
  $(x,y)_1 = (0,5)$
  $(x,y)_2 = (1,3)$
  $(x,y)_3 = (2,1)$
  $(x,y)_4 = (3,-1)$
  $(x,y)_5 = (4,-3)$
  etc.

- **Addition** of two equations of the system of equations

  Two equations of a system of equations can be transformed into one single equation by adding both the left hand sides and the right hand sides of the equations. The solutions of the new equation contain sets of numbers that are solutions of both of the two original equations (without proof).

  Ex.: $2x + y = 5 \quad$ I
  $x + 2y = 4 \quad$ II

  Adding both the left hand sides and the right hand sides of the two equations yields a new equation

  $3x + 3y = 9 \quad$ III

  Equation III has the solutions
  $(x,y)_1 = (0,3)$
  $(x,y)_2 = (1,2)$
  $(x,y)_3 =$ **(2,1)**
  $(x,y)_4 = (3,0)$
  etc.
  These solutions contain the set (x,y) = (2,1) which is a solution of both of the original equations I and II.

2. Strategy for solving a system of n equations in n variables

Both the number of equations and the number of variables can be reduced by applying appropriate operations (see 1. above) to the system of equations:

System of n equations in n variables
→ *System of (n-1) equations in (n-1) variables*
→ *System of (n-2) equations in (n-2) variables*
...
→ *System of 2 equations in 2 variables*
→ *1 equation in 1 variable*

3. Solving a linear system of equations

- **Addition method**

Ex.: $4x + 7y = -16$ I
$7x - 3y = 33$ II
(2 equations in 2 variables)

Finding appropriate multiples of both I and II

3·I $12x + 21y = -48$ III
7·II $49x - 21y = 231$ IV

Adding III and IV and solving for x

III+IV $61x = 183$ | : 61
(1 equation in 1 variable)

$x = 3$

Substituting 3 for x in I and solving for y

$4 \cdot 3 + 7y = -16$ | - 12
$7y = -28$ | : 7
$y = -4$

(x,y) = (3,-4)

- **Equation method**

Ex.: $4x + 7y = -16$ I
$7x - 3y = 33$ II
(2 equations in 2 variables)

Solving I for x

$4x + 7y = -16$ | - 7y
$4x = -7y - 16$ | : 4
$x = \frac{-7y - 16}{4}$ III

Solving II for x

$7x - 3y = 33$ | + 3y
$7x = 3y + 33$ | : 7
$x = \frac{3y + 33}{7}$ IV

Equating expressions for x in III und IV and solving for y

$\frac{-7y - 16}{4} = \frac{3y + 33}{7}$ | · 28
(1 equation in 1 variable)

$7(-7y - 16) = 4(3y + 33)$
$-49y - 112 = 12y + 132$ | + 49y | - 132
$61y = -244$ | : 61
$y = -4$

Substituting -4 for y in III

$x = \frac{-7(-4) - 16}{4} = 3$

(x,y) = (3,-4)

- **Substitution method**

Ex.: $4x + 7y = -16$ I
$7x - 3y = 33$ II
(2 equations in 2 variables)

Solving I for x

$4x + 7y = -16 \qquad | -7y$
$4x = -7y - 16 \qquad | : 4$
$x = \frac{-7y - 16}{4}$ III

Substituting the value of x for x in II and solving for y

$7\,\frac{-7y - 16}{4} - 3y = 33 \qquad | \cdot 4$
(1 equation in 1 variable)

$7(-7y - 16) - 12y = 132$
$-49y - 112 - 12y = 132$
$-61y - 112 = 132 \qquad | + 112$
$-61y = 244 \qquad | : (-61)$
$y = -4$

Substituting -4 for y in III

$x = \frac{-7(-4) - 16}{4} = 3$

(x,y) = (3,-4)