Exercise 4 Linear function and equations
Linear function, simple interest, cost, revenue, profit, break-even

Objectives

- be able to think of a relation between two quantities as a function.
- be able to determine the domain, the codomain, the range of a given function.
- be able to draw the graph of a given linear function.
- be able to determine slope and intercept of a linear function.
- know some examples of linear functions in economic and everyday life applications.
- know and understand the term "simple interest".
- be able to perform simple interest calculation.
- know and understand the terms "fixed costs", "variable costs", "total cost", "total revenue", "total profit" and "break-even value".
- be able to apply the concept of linear functions to a new problem.

Problems

1. A taxi driver charges the following fare:

   8 CHF plus 1 CHF per kilometer

   Think of the taxi fare as a function f.

   a) Determine the domain D, the codomain B, and the range E of the function.

   b) Draw the graph of the function f.

2. The taxi fare as described in problem 1 can be thought of as a linear function which assigns to each distance x (in km) a fare y (in CHF):

   $$f: \mathbb{R}_0^+ \to \mathbb{R}_0^+$$
   $$x \to y = f(x) = ax + b$$

   Determine the values of a and b.

3. Find at least two more examples of linear functions in economics or in an everyday life context.

4. Graph the linear functions below, and determine both slope and intercept:

   a) $f: \mathbb{R} \to \mathbb{R}$, $x \to y = f(x) = -2$

   b) $f: \mathbb{R} \to \mathbb{R}$, $x \to y = f(x) = 3x-4$

   c) $f: \mathbb{R} \to \mathbb{R}$, $x \to y = f(x) = -x+3$

5. Simple interest at a rate of 0.5% is paid on an initial bank balance of 5000 CHF.

   a) Determine the balance after ten years' time.

   b) Determine both slope and intercept of the corresponding linear function.

6. A mobile phone company offers three different tariffs:

Tariff A:	monthly basic fee of 10 CHF plus 0.20 CHF per minute
Tariff B:	monthly basic fee of 25 CHF plus 0.10 CHF per minute
Tariff C:	no basic fee, 0.60 CHF per minute

Think of the the three tariffs as linear functions.

a) Draw the graphs of the three functions in one common coordinate system.

b) Determine the total fee for each tariff for a monthly phone call duration of 1 hour.

c) For what monthly phone call duration tariff A is cheaper than tariff C?

d) For what monthly phone call duration tariff B is cheaper than tariff A?

7. (from: Bittinger, Ellenbogen: Calculus and its applications, Pearson 2007, ISBN 0-321-48543-2)

EXAMPLE 9 Business: Total Cost. Raggs, Ltd., a clothing firm, has **fixed costs** of $10,000 per year. These costs, such as rent, maintenance, and so on, must be paid no matter how much the company produces. To produce x units of a certain kind of suit, it costs $20 per suit (unit) in addition to the fixed costs. That is, the **variable costs** for producing x of these suits are $20x$ dollars. These costs are due to the amount produced and stem from items such as material, wages, fuel, and so on. The **total cost** $C(x)$ of producing x suits in a year is given by a function C:

$$C(x) = (\text{Variable costs}) + (\text{Fixed costs}) = 20x + 10{,}000.$$

a) Graph the variable-cost, the fixed-cost, and the total-cost functions.

b) What is the total cost of producing 100 suits? 400 suits?

8. (from: Bittinger, Ellenbogen: Calculus and its applications, Pearson 2007, ISBN 0-321-48543-2)

EXAMPLE 10 Business: Profit-and-Loss Analysis. When a business sells an item, it receives the *price* paid by the consumer (this is normally greater than the *cost* to the business of producing the item).

a) The **total revenue** that a business receives is the product of the number of items sold and the price paid per item. Thus, if Raggs, Ltd., sells x suits at $80 per suit, the total revenue $R(x)$, in dollars, is given by

$$R(x) = \text{Unit price} \cdot \text{Quantity sold} = 80x.$$

If $C(x) = 20x + 10{,}000$ (see Example 9), graph R and C using the same set of axes.

b) The **total profit** that a business receives is the amount left after all costs have been subtracted from the total revenue. Thus, if $P(x)$ represents the total profit when x items are produced and sold, we have

$$P(x) = (\text{Total revenue}) - (\text{Total costs}) = R(x) - C(x).$$

Determine $P(x)$ and draw its graph using the same set of axes as was used for the graph in part (a).

c) The company will *break even* at that value of x for which $P(x) = 0$ (that is, no profit and no loss). This is the point at which $R(x) = C(x)$. Find the **break-even value** of x.

Answers

1. a) $D = \mathbb{R}_0^+$ (distance/km)
$B = \mathbb{R}_0^+$ (fare/CHF)
$E = \{y \in \mathbb{R}_0^+ \mid y \geq 8\}$

b) ...

2. $a = 1$, $b = 8$

3. ...

4. a) Slope $a = 0$, intercept $b = -2$
b) Slope $a = 3$, intercept $b = -4$
c) Slope $a = -1$, intercept $b = 3$

5. a) 5250 CHF

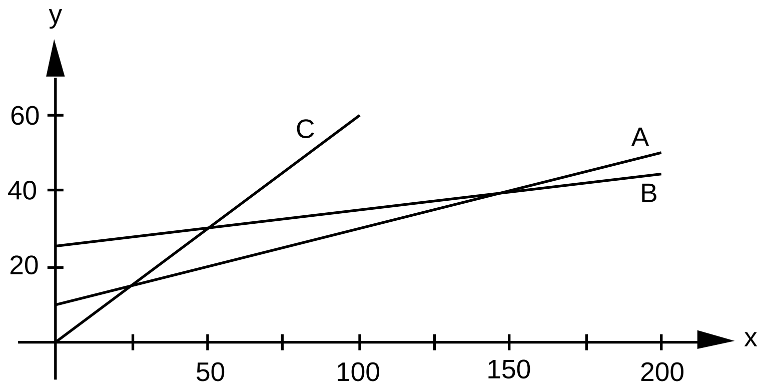
b) $f: \mathbb{R}_0^+ \to \mathbb{R}_0^+$
$x \to y = f(x) = ax + b$

Slope $a = 25$, intercept $b = 5000$

6. a) Tariff A: $A: \mathbb{R}_0^+ \to \mathbb{R}_0^+$
$x \to y = A(x) = 0.2\,x + 10$

Tariff B: $B: \mathbb{R}_0^+ \to \mathbb{R}_0^+$
$x \to y = B(x) = 0.1\,x + 25$

Tariff C: $C: \mathbb{R}_0^+ \to \mathbb{R}_0^+$
$x \to y = C(x) = 0.6\,x$
Direct proportionality: fee y is direct proportional to phone call duration x.



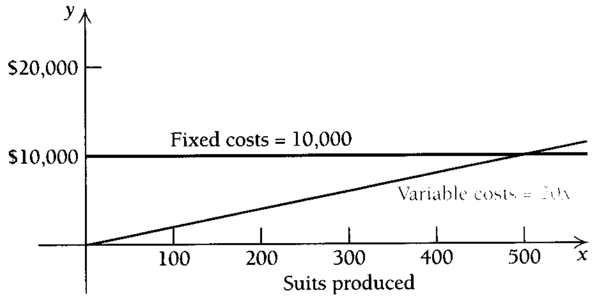
b) Tariff A: 22 CHF
Tariff B: 31 CHF
Tariff C: 36 CHF

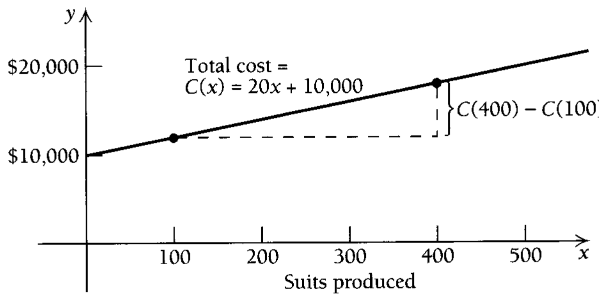
c) over 25 min

d) over 150 min

7.

a) The variable-cost and fixed-cost functions appear in the graph on the left below. The total-cost function is shown in the graph on the right. From a practical standpoint, the domains of these functions are nonnegative integers 0, 1, 2, 3, and so on, since it does not make sense to make either a negative number or a fractional number of suits. It is common practice to draw the graphs as though the domains were the entire set of nonnegative real numbers.





b) The total cost of producing 100 suits is

$$C(100) = 20 \cdot 100 + 10{,}000 = \$12{,}000.$$

The total cost of producing 400 suits is

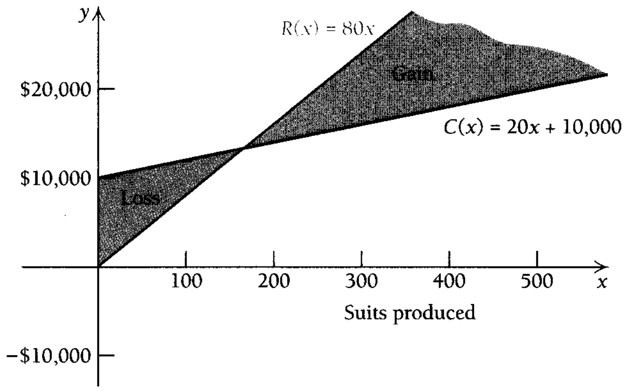
$$\begin{aligned} C(400) &= 20 \cdot 400 + 10{,}000 \\ &= \$18{,}000. \end{aligned}$$

✦

8. (see page 5)

8.

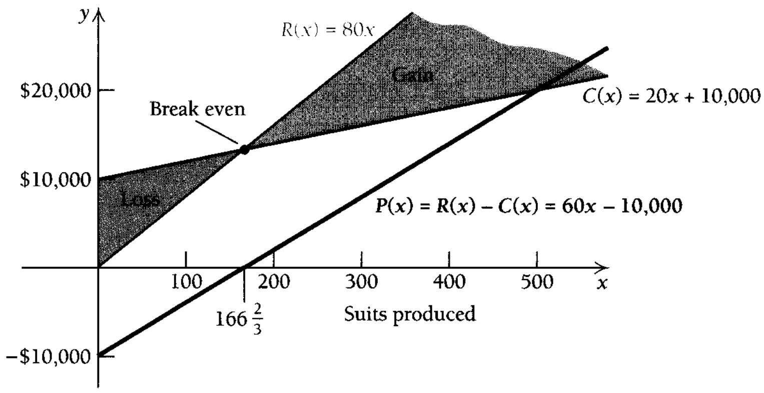
a) The graphs of $R(x) = 80x$ and $C(x) = 20x + 10{,}000$ are shown below. When $C(x)$ is above $R(x)$, a loss will occur. This is shown by the region shaded red. When $R(x)$ is above $C(x)$, a gain will occur. This is shown by the region shaded gray.



b) To find P, the profit function, we have

$$P(x) = R(x) - C(x) = 80x - (20x + 10{,}000)$$
$$= 60x - 10{,}000.$$

The graph of $P(x)$ is shown by the heavy line. The red portion of the line shows a "negative" profit, or loss. The black portion of the heavy line shows a "positive" profit, or gain.



c) To find the break-even value, we solve $R(x) = C(x)$:

$$R(x) = C(x)$$
$$80x = 20x + 10{,}000$$
$$60x = 10{,}000$$
$$x = 166\tfrac{2}{3}.$$

How do we interpret the fractional answer, since it is not possible to produce $\frac{2}{3}$ of a suit? We simply round to 167. Estimates of break-even values are usually sufficient since companies want to operate well away from break-even values in order to maximize profit. ✦