Review exercises 1 — Functions and equations

Problems

R1.1 Decide which of the following relations are functions. Give reasons for your answers.

a) $f_1 : \mathbb{R}_0^+ \to \mathbb{R}^+, x \mapsto y = f_1(x) = \sqrt{x}$

b) $f_2 : \{2, 3, 4, ...\} \to \mathbb{N}, x \mapsto y = f_2(x) = x - 1$

c) D = Set of all Swiss cantons
C = Set of all Swiss towns and cities
$f_3 : D \to C, x \mapsto y = f_3(x) =$ capital of x

d) $f_4 : \{x: x \in \mathbb{R} \text{ and } x \geq 3\} \to \mathbb{R}, x \mapsto y = f_4(x) = \frac{1}{x^2 - 9}$

e) $f_5 : \mathbb{R}_0^+ \to \mathbb{R}, x \mapsto y = f_5(x) = \log_a(x)$

R1.2 If $f(x) = 9x - x^2$, find ...

a) ... f(0).

b) ... f(-3).

c) ... $\frac{f(x + h) - f(x)}{h}$ and simplify the expression.

R1.3 Solve the equations below, and state the solution sets:

a) $3x - 8 = 23$

b) $\frac{6}{3x - 5} = \frac{6}{2x + 3}$

c) $\frac{2x + 5}{x + 7} = \frac{1}{3} + \frac{x - 11}{2x + 14}$

R1.4 Solve the following equations for x, and state the solution sets.
Take into account that the parameters a and p can be any real numbers.

a) $ax = 60$

b) $(p - 1)px = p^2 - 1$

R1.5 Solve each system of equations, and state the solution sets:

a) $2x + y = 19$
$x - 2y = 12$

b) $6x + 3y = 1$
$y = - 2x + 1$

R1.6 Determine the equation of the linear function whose graph ...

a) ... has slope 4 and intercept 2.

b) ... passes through (-2|1) and has slope $\frac{2}{5}$.

c) ... (see next page)

c) ... passes through (-2|7) and (6|-4).

d) ... passes through (1|6) and is parallel to $y = 4x - 6$.

R1.7 A certain product has the following supply and demand functions:

$p = f_s(q) = (4q + 5)$ CHF
$p = f_d(q) = (-2q + 81)$ CHF

a) Determine both the number of units supplied and the number of units demanded if the price is 53 CHF.

b) Determine both the equilibrium quantity and the equilibrium price.

R1.8 The total cost and total revenue for a certain product are given by the following:

$C(x) = (38.80x + 4500)$ CHF
$R(x) = 61.30x$ CHF

a) Determine the fixed costs.

b) Determine the variable costs for producing 10 units.

c) Determine the number of units required to break even.

R1.9 The supply and the demand functions for a product are linear and determined by the tables below:

Supply function		**Demand function**	
Price	**Quantity**	**Price**	**Quantity**
100 CHF	200	200 CHF	200
200 CHF	400	100 CHF	400
300 CHF	600	0 CHF	600

Determine the quantity and price that will give market equilibrium.

R1.10 Determine the solution sets of the equations below:

a) $4x - 3x^2 = 0$

b) $3x^2 - 6x = 9$

c) $4x^2 + 25 = 0$

d) $\frac{1}{x} + 2x = \frac{1}{3} + \frac{x+1}{x}$

e) $\frac{x-4}{x-5} = \frac{30-x^2}{x^2-5x}$

R1.11 Determine the equation of the quadratic function whose graph ...

a) ... has the vertex (2|4) and passes through (3|3).

b) ... passes through (-3|-3), (0|3), and (3|0).

R1.12 The supply function for a product is given by $p = (q^2 + 300)$ CHF, and the demand is given by $p/\text{CHF} + q = 410$.

Determine the equilibrium quantity and price.

R1.13 The total costs for a service (x = service quantity) are given by $C(x) = (1760 + 8x + 0.6x^2)$ CHF and the total revenues are given by $R(x) = (100x - 0.4x^2)$ CHF.

Determine the break-even points.

R1.14 Determine the equation of the exponential function whose graph passes through P and Q.

a) P(0|1) Q(2|9)

b) P(1|20) Q(2|100)

R1.15 Evaluate each logarithm without using a calculator:

a) $\log_5(1)$

b) $\log_2(8)$

c) $\log_3\left(\frac{1}{3}\right)$

d) $\log_3(3^8)$

e) $e^{\ln(5)}$

f) $10^{\lg(3.15)}$

R1.16 8000 CHF are borrowed for 3 years at an annual simple interest rate of 12%.

Determine the future value of the loan at the end of the 3 years.

R1.17 Mary borrowed 2000 CHF from her parents and repaid them 2200 CHF after 4 years.

Determine the annual simple interest rate at which Mary borrowed the money.

R1.18 A future student wants to deposit money on a bank account in order to have 3000 CHF for tuition and fees 3 years later. The investement pays simple interest at an annual interest rate of 6%.

Determine the amount of many the future student must pay in.

R1.19 Assume that 1000 CHF are invested for 4 years at a nominal annual interest rate of 8%, compounded quarterly.

Determine the amount of interest that will be earned in the 4 years.

R1.20 An investment pays interest at a nominal annual interest rate of 5.4%, compounded monthly. The goal is to have 18'000 CHF in 4 years.

Determine the amount of money that has to be invested.

R1.21 In 2010 an African country had a population of 4.5 million. The population has been increasing at 4% per year.

Determine what the population will be in 2030 if the growth factor does not change.

R1.22 (see next page)

R1.22 A company wants to have 250'000 CHF available in 4 1/2 years for new construction. The investement pays interest at a nominal annual interest rate of 10.2%, compounded quarterly.

Determine how much must be deposited at the beginning of each quarter to reach the mentioned goal.

R1.23 A retirement account that pays interest at a nominal annual interest rate of 6.8%, compounded semiannually, contains 488'000 CHF.

Determine how long 40'000 CHF can be withdrawn at the end of each half-year until the account balance is 0 CHF.

R1.24 Three years from now, a couple plan to spend 4 months travelling in China, Japan, and Southeast Asia. When they take their trip, they would like to withdraw 5000 CHF at the beginning of each month to cover their expenses for that month. Assume that the account pays interest at a nominal annual interest rate of 6.6%, compounded monthly.

Determine how much the couple must deposit at the beginning of each month for the next 3 years so that there is enough money in their account when they start travelling.

R1.25 Mr S. is obligated to pay 25'000 CHF at the end of each of the following 8 years to his divorced wife. As a result of a personal profit in his company, he is able to pay the whole sum at the end of the first year (instead of making 8 payments at the end of each year). Assume that interest, compounded annually, has been fixed at an annual interest rate of 4.5%.

Determine the amount of money Mr. S. has to pay at the end of the first year.

R1.26 Mr P. is thinking about an investement for his retirement. He would like to withdraw 8000 CHF from an account at the end of each year for 15 years starting at the end of the year in which he turns 60. He assumes that money pays interest at an annual interest rate of 2.5% throughout the 15 years.

a) He wants to save the money by making 30 constant payments at the end of each year until turning 55. He assumes that his bank pays him interest at an annual interest rate of 3%, compounded annually, and that the interest rate of 3% is valid until he turns 60.

Determine how much Mr P. must pay in each year.

b) Mr P. has won 40'000 CHF in a lottery! Assume that he pays the money in at the end of the year in which he turns 25 and that there are the same interest conditions as in a).

Decide whether this amount is sufficient for his retirement scheme. Give reasons for your answer.

Answers

R1.1 a) no function
f is not defined for x = 0 (although $0 \in \mathbb{R}_0^+$), as $y = f(0) = 0 \notin \mathbb{R}^+$

b) function

c) function

d) no function
f is not defined for x = 3.

e) no function
f is not defined for x = 0.

Hints:
- A function must be defined for each element of the domain.
- A function must be unique, i.e. only one element of the codomain is assigned to each element in the domain.

R1.2 a) f(0) = 0

b) f(-3) = -36

c) $\frac{f(x + h) - f(x)}{h} = 9 - 2x - h$

R1.3 a) $S = \left\{\frac{31}{3}\right\}$

b) S = {8}

Hint:
- First get rid of the fraction by multiplying by the least common denominator.

c) S = {}

Hints:
- Use the same procedure as in b).
- Because of the denominators x + 7 and 2x + 14 in the original equation, -7 cannot be a solution.

R1.4 a)

if $a = 0$:	no solution	⇒	$S = \{\ \}$
if $a \neq 0$:	$x = \frac{60}{a}$	⇒	$S = \left\{\frac{60}{a}\right\}$

b)

if $p = 0$:	no solution	⇒	$S = \{\ \}$
if $p = 1$:	$x \in \mathbb{R}$	⇒	$S = \mathbb{R}$
if $p \neq 0$ and $p \neq 1$:	$x = \frac{p+1}{p}$	⇒	$S = \left\{\frac{p+1}{p}\right\}$

Hints:
- Divison by 0 is not defined.
- A division by a number that contains the parameter a or p requires a case differentiation.

R1.5 a) (x, y) = (10, -1)
S = {(10, -1)}

b) S = { }

Hints:
- First solve one equation for y (or x).
- Substitute the expression for y (or x) in the other equation.
- Solve the equation for x (or y).

R1.6 a) $y = f(x) = 4x + 2$

b) $y = f(x) = \frac{2}{5}x + \frac{9}{5}$

c) $y = f(x) = -\frac{11}{8}x + \frac{17}{4}$

d) $y = f(x) = 4x + 2$

Hints:
- First state the general form of the equation of a linear function.
- Determine the two parameters (a and b) of the equation by building up a system of two equations according to the stated problem.
- A point is on the graph of a function if and only if its coordinates fulfil the equation of the function.

R1.7 a) 12 supplied, 14 demanded

b) $f_s(q) = f_d(q)$ for $q = \frac{38}{3} = 12.6... \notin \mathbb{N}$
$\Rightarrow$ no exact equilibrium $\rightarrow$ $q = 13$, $f_s(13) = 57$ CHF, $f_d(13) = 55$ CHF

R1.8 a) 4500 CHF

b) 388 CHF

c) $C(x) = R(x)$ for $x = 200$

R1.9 Supply function: $f_s(q) = \frac{1}{2}q$ CHF

Demand function: $f_d(q) = \left(-\frac{1}{2}q + 300\right)$ CHF

Market equilibrium: $f_s(q) = f_d(q)$ for $q = 300$ and $p = 150$ CHF

R1.10 a) $S = \{0, 4/3\}$

Hints:
- Factorise the left hand side of the equation (factor x).
- A product is equal to 0 if and only if at least one factor is equal to 0.

b) $S = \{-1, 3\}$

Hint:
- Use the quadratic formula.

c) $S = \{\ \}$

Hints:
- First solve for x^2.
- The square of any real number is equal to or greater than 0.

d) $S = \{2/3\}$

Hints:
- First get rid of the fractions by multiplying by the least common denominator (= 3x).
- The fractions $\frac{1}{x}$ and $\frac{x+1}{x}$ are not defined for x = 0. Hence, x = 0 cannot be a solution.

e) (see next page)

e) (same equation as in exercise 7.6 c))

$S = \{-3\}$

Hints:
- First get rid of the fractions by multiplying by the least common denominator (= x (x - 5)).
- The fractions in the original equation are not defined for x = 5. Hence, x = 5 cannot be a solution.

R1.11 a) $y = f(x) = -(x-2)^2 + 4$

b) $y = f(x) = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$

Hints:
- First state the equation of a general quadratic function.
- Use the vertex form of the equation in a).
- Use the general form of the equation in b).
- Determine the parameters in the equation by building up a system of equations according to the stated problem.
- A point is on the graph of a function if and only if its coordinates fulfil the formula of the function.

R1.12 Supply function: $f_s(q) = (q^2 + 300)$ CHF
Demand function: $f_d(q) = (-q + 410)$ CHF

Market equilibrium: $f_s(q) = f_d(q)$ for q = 10 and p = 400 CHF

R1.13 C(x) = R(x)
$x_1 = 46 + 2\sqrt{89} = 64.9$ (rounded)
$x_2 = 46 - 2\sqrt{89} = 27.1$ (rounded)

R1.14 a) $y = f(x) = 3^x$

b) $y = f(x) = 4 \cdot 5^x$

Hints:
- First state the equation of a general exponential function.
- Determine the parameters in the equation by building up a system of equations according to the stated problem.
- A point is on the graph of a function if and only if its coordinates fulfil the equation of the function.

R1.15 a) 0

b) 3

c) - 1

d) 8

Hint:
- The expression $\log_a(x)$ is the answer to the question "a to what power is equal to x?"

e) 5

f) 3.15

Hint:
- Use that $a^{\log_a(x)} = x$ for any $a \in \mathbb{R}^+ \setminus \{1\}$.

R1.16 Simple interest

$C_n = C_0(1 + nr)$ where C_0 = 8000 CHF, r = 12%, n = 3

$\Rightarrow$ C_3 = 10'880 CHF

R1.17 Simple interest

$r = \frac{\frac{C_n}{C_0} - 1}{n}$ where C_0 = 2000 CHF, C_n = 2200 CHF, n = 4

$\Rightarrow$ r = 2.5%

R1.18 Simple interest

$C_0 = \frac{C_n}{1 + nr}$ where C_n = 3000 CHF, r = 6%, n = 3

$\Rightarrow$ C_0 = 2542.37 CHF (rounded)

R1.19 Compound interest

$C_n = C_0 (1 + r)^n$ where C_0 = 1000 CHF, $r = \frac{8\%}{4} = 2\%$, n = 4·4 = 16

$\Rightarrow$ $C_n - C_0$ = 372.79 CHF (rounded)

R1.20 Compound interest

$C_0 = \frac{C_n}{(1 + r)^n}$ where C_n = 18'000 CHF, $r = \frac{5.4\%}{12}$, n = 12·4 = 48

$\Rightarrow$ C_0 = 14'510.26 CHF (rounded)

R1.21 9.86 million (rounded)

Hints:
- The population grows exponentially.
- State the general form of the exponential function.
- Find out both the initial value and the growth factor.

More detailed answer:
- $y = f(x) = c \cdot a^x$
- initial value (population in 2010): c = f(0) = 4'500'000
- growth factor a = 1 + 4% = 1.04
- population in 2030: $f(20) = 4'500'000 \cdot 1.04^{20}$ = 9.86 Mio (rounded)

R1.22 Annuity due

$p = \frac{A_n(q - 1)}{q(q^n - 1)}$ where A_n = 250'000 CHF, $q = 1 + \frac{10.2\%}{4}$, n = 4.5·4 = 18

$\Rightarrow$ p = 10'841.24 CHF (rounded)

R1.23 (see next page)

R1.23 Ordinary annuity

$n = \frac{\lg\left(\frac{p}{p - A_0(q - 1)}\right)}{\lg(q)}$ where A_0 = 488'000 CHF, p = 40'000 CHF, $q = 1 + \frac{6.8\%}{2}$

⇒ n = 16.02... → 16 half-years = 8 years

R1.24 2 annuities: 3 years starting from now (paying in money), 4 months (withdrawing money)

- 4 months (withdrawing money): annuity due

$A_0 = p\,\frac{q^n - 1}{q^{n-1}(q - 1)}$ where p = 5000 CHF, $q = 1 + \frac{6.6\%}{12}$, n = 4

⇒ A_0 = 19'836.49… CHF

- 3 years starting from now (paying in money): annuity due

$p = \frac{A_n(q - 1)}{q(q^n - 1)}$ where A_n = … (= A_0 in first annuity), $q = 1 + \frac{6.6\%}{12}$, n = 36

⇒ p = 497.04 CHF (rounded)

R1.25 The whole sum Mr S. pays in at the end of the first year pays interest. The capital at the end of the 8th year must be the same as the value the annuity would have if Mr S. made 8 payments at the end of each year.

- Ordinary annuity

$A_n = p\,\frac{q^n - 1}{q - 1}$ where p = 25'000 CHF, q = 1 + 4.5%, n = 8

⇒ A_n = 234'500.34… CHF

- Compound interest

$C_0 = \frac{C_n}{(1 + r)^n}$ where C_n = … (= A_n in annuity), r = 4.5%, n = 7

⇒ C_0 = 172'317.53 CHF (rounded up)

R1.26 a) - Ordinary annuity (from age 60 to age 75)

$A_0 = p\,\frac{q^n - 1}{q^n(q - 1)}$ where p = 8000 CHF, q = 1 + 2.5%, n = 15

⇒ A_0 = 99'051.02… CHF

- Compound interest (from age 55 to age 60)

$C_0 = \frac{C_n}{(1 + r)^n}$ where C_n = … (= A_0 in annuity from age 60 to age 75), r = 3%, n = 5

⇒ C_0 = 85'442.28… CHF

- Ordinary annuity (from age 25 to age 55)

$p = \frac{A_n(q - 1)}{q^n - 1}$ where A_n = … (= C_0), q = 1 + 3%, n = 30

⇒ p = 1795.93 CHF

b) Compound interest (from age 25 to age 60)

$C_n = C_0\,(1 + r)^n$ where C_0 = 40'000 CHF, r = 3%, n = 35

⇒ C_n = 112'554.50 CHF (rounded) > … (= A_0 in annuity from age 60 to age 75)

⇒ The amount is sufficient for his retirement scheme.