Exercises 15 Definite integral
Definite integral, area under a curve, consumer's/producer's surplus

Objectives

- be able to apply the fundamental theorem of calculus.
- be able to determine a definite integral of a constant, basic power, and basic exponential function.
- be able to determine the area between the graph of a basic power function and the abscissa.
- be able to determine a consumer's and a producer's surplus if the demand and supply functions are basic power functions.

Problems

15.1 Calculate the definite integrals below:

a) $\int_3^4 (2x - 5)\, dx$ b) $\int_0^1 (x^3 + 2x)\, dx$ c) $\int_{-5}^{-3} \left(\frac{1}{2}x^2 - 4\right) dx$

d) $\int_2^4 \left(x^3 - \frac{1}{2}x^2 + 3x - 4\right) dx$ e) $\int_{-2}^{2} \left(-\frac{1}{8}x^4 + 2x^2\right) dx$ f) $\int_{-1}^{1} e^x\, dx$

g) $\int_0^1 e^{2x}\, dx$ h) $\int_{-1}^{1} e^{-3x}\, dx$

15.2 Determine the area between the graph of the function f and the x-axis on the interval where the graph of f is above the x-axis, i.e. where $f(x) \geq 0$.

a) $f(x) = -x^2 + 1$ b) $f(x) = x^3 - x^2 - 2x$

Hints:
- First, determine the positions x where the graph of f touches or intersects the x-axis, i.e where $f(x) = 0$
- Then, determine the interval on which the graph of f is above the x-axis, i.e. where $f(x) \geq 0$

15.3 The demand function for a product is $p = f_d(x) = (100 - 4x^2)$ CHF. The equilibrium quantity is 4 units.

Determine the consumer's surplus.

15.4 The demand function for a product is $p = f_d(x) = (34 - x^2)$ CHF. The equilibrium price is 9 CHF.

Determine the consumer's surplus.

15.5 Suppose that the supply function for a good or a service is $p = f_s(x) = (4x^2 + 2x + 2)$ CHF, and that the equilibrium price is 422 CHF.

Determine the producer's surplus.

15.6 The the supply function f_s and the demand function f_d for a certain product or service are given as follows:

$p = f_s(x) = (x^2 + 4x + 11)$ CHF

$p = f_d(x) = (81 - x^2)$ CHF

Determine …

a) … the equilibrium point, i.e. the equilibrium quantitiy and the equilibrium price.

b) … the consumer's surplus.

c) … the producer's surplus.

15.7 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) The definite integral of a function is a ...

- ☐ ... real number.
- ☐ ... function.
- ☐ ... set of functions.
- ☐ ... graph.

b) $\int_a^b f(x)\,dx$ ...

- ☐ ... = f(b) - f(a)
- ☐ ... = F(a) - F(b) where F is an antiderivative of f.
- ☐ ... is equal to the area between the graph of f and the x-axis on the interval $a \leq x \leq b$ if $f(x) \geq 0$ on the interval $a \leq x \leq b$.
- ☐ ... cannot be calculated unless all antiderivatives of f are known.

c) The consumer's surplus is an area between ...

- ☐ ... the graphs of the demand and the supply functions.
- ☐ ... the x axis and the graph of the demand function.
- ☐ ... the graph of the demand function and the horizontal line "price = equilibrium price".
- ☐ ... the horizontal line "price = equilibrium price" and the graph of the supply function.

Answers

15.1 a) $\int_3^4 (2x - 5)\,dx = \left[2\cdot\frac{1}{2}x^2 - 5x\right]_3^4 = [x^2 - 5x]_3^4 = (4^2 - 5\cdot 4) - (3^2 - 5\cdot 3) = 2$

b) $\int_0^1 (x^3 + 2x)\,dx = \left[\frac{1}{4}x^4 + 2\cdot\frac{1}{2}x^2\right]_0^1 = \left[\frac{1}{4}x^4 + x^2\right]_0^1 = \left(\frac{1}{4}1^4 + 1^2\right) - \left(\frac{1}{4}0^4 + 0^2\right) = \frac{5}{4}$

c) $\int_{-5}^{-3} \left(\frac{1}{2}x^2 - 4\right) dx = \left[\frac{1}{2}\cdot\frac{1}{3}x^3 - 4x\right]_{-5}^{-3} = \left[\frac{1}{6}x^3 - 4x\right]_{-5}^{-3} = \left(\frac{1}{6}(-3)^3 - 4\cdot(-3)\right) - \left(\frac{1}{6}(-5)^3 - 4\cdot(-5)\right) = \frac{25}{3}$

d) $\int_2^4 \left(x^3 - \frac{1}{2}x^2 + 3x - 4\right) dx = \left[\frac{1}{4}x^4 - \frac{1}{2}\cdot\frac{1}{3}x^3 + 3\cdot\frac{1}{2}x^2 - 4x\right]_2^4 = \left[\frac{1}{4}x^4 - \frac{1}{6}x^3 + \frac{3}{2}x^2 - 4x\right]_2^4$
$= \left(\frac{1}{4}4^4 - \frac{1}{6}4^3 + \frac{3}{2}4^2 - 4\cdot 4\right) - \left(\frac{1}{4}2^4 - \frac{1}{6}2^3 + \frac{3}{2}2^2 - 4\cdot 2\right) = \frac{182}{3}$

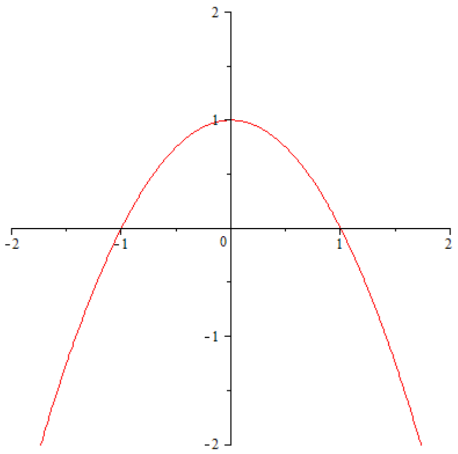
e) $\int_{-2}^2 \left(-\frac{1}{8}x^4 + 2x^2\right) dx = \left[-\frac{1}{8}\cdot\frac{1}{5}x^5 + 2\cdot\frac{1}{3}x^3\right]_{-2}^2 = \left[-\frac{1}{40}x^5 + \frac{2}{3}x^3\right]_{-2}^2$
$= \left(-\frac{1}{40}2^5 + \frac{2}{3}2^3\right) - \left(-\frac{1}{40}(-2)^5 + \frac{2}{3}(-2)^3\right) = \frac{136}{15}$

f) $\int_{-1}^1 e^x\,dx = [e^x]_{-1}^1 = e^1 - e^{-1} = e - \frac{1}{e}$

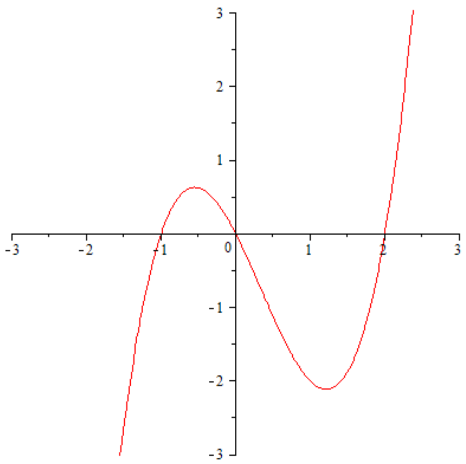
g) $\int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}[e^{2x}]_0^1 = \frac{1}{2}(e^{2\cdot 1} - e^{2\cdot 0}) = \frac{1}{2}(e^2 - 1)$

h) $\int_{-1}^1 e^{-3x}\,dx = \left[-\frac{1}{3}e^{-3x}\right]_{-1}^1 = -\frac{1}{3}[e^{-3x}]_{-1}^1 = -\frac{1}{3}\left(e^{-3\cdot 1} - e^{-3\cdot(-1)}\right) = -\frac{1}{3}(e^{-3} - e^3) = \frac{1}{3}\left(e^3 - \frac{1}{e^3}\right)$

15.2 a) $A = \int_{-1}^1 (-x^2 + 1)\,dx = \left[-\frac{1}{3}x^3 + x\right]_{-1}^1 = \frac{4}{3}$



b) $A = \int_{-1}^0 (x^3 - x^2 - 2x)\,dx = \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2\cdot\frac{1}{2}x^2\right]_{-1}^0 = \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2\right]_{-1}^0 = \frac{5}{12}$



15.3 Consumer's surplus
CS = 170.67 CHF (rounded)

15.4 Consumer's surplus
CS = 83.33 CHF (rounded)

15.5 Producer's surplus
PS = 2766.67 CHF (rounded)

15.6 a) Equilibrium quantity
x = 5

Equilibrium price
p = 56 CHF

b) Consumer's surplus
CS = 83.33 CHF (rounded)

c) Producer's surplus
PS = 133.33 CHF (rounded)

15.7 a) 1^{st} statement

b) 3^{rd} statement

c) 3^{rd} statement