Exercises 13 Applications of differential calculus
Local/global maxima/minima, points of inflection

Objectives

- be able to determine the local maxima and minima of a function.
- be able to determine the points of inflection of a function.
- be able to determine the global maximum and the global minimum of a cost, revenue, and profit function.
- be able to determine the global minimum of an average cost, average revenue, and average profit function.

Problems

13.1 Determine all positions where the given function has …

i) … a local maximum.
… a local minimum.

ii) … a point of inflection.

a) $f(x) = x^2 - 4$

b) $f(x) = - 8x^3 + 12x^2 + 18x$

c) $s(t) = t^4 - 8t^2 + 16$

d) $f(x) = x\, e^{-x}$

e) * $f(x) = (1 - e^{-2x})^2$

f) * $V(r) = - D\left(\frac{2a}{r} - \frac{a^2}{r^2}\right)$ $(D > 0, a > 0)$

13.2 If the total profit for a commodity is

$$P(x) = (2000x + 20x^2 - x^3) \text{ CHF}$$

where x is the number of items sold, determine the level of sales, x, that maximises profit, and find the maximum profit.

Hints:
- First, find the local maxima.
- Then, check if one of the local maxima is the global maximum.

13.3 The total cost for a service concerning a tourism event is given by

$$C(x) = \left(\frac{1}{4}x^2 + 4x + 100\right) \cdot 100 \text{ CHF}$$

where x represents the extent of the service.

Determine the value of x that will result in a minimum average cost.

Determine that minimum average cost, too.

13.4 Suppose that the production capacity for a certain commodity cannot exceed 30. The total profit for this company is

$$P(x) = (4x^3 - 210x^2 + 3600x) \text{ CHF}$$

where x is the number of units sold.

Determine the number of units that will maximise profit.

13.5 Suppose the annual profit for a store is given by

$$P(x) = (-0.1x^3 + 3x^2) \cdot 1000 \text{ CHF}$$

where x is the number of years past 2010.

Determine the point of inflection for the profit.

13.6 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) If f has a local maximum at x_0 it can be concluded that ...

- ☐ ... $f(x_0) > f(x)$ for any $x \neq x_0$
- ☐ ... $f(x_0) > f(x)$ for any $x > x_0$
- ☐ ... $f(x_0) > f(x)$ for any $x < x_0$
- ☐ ... $f(x_0) > f(x)$ for all x which are in a certain neighbourhood of x_0

b) If $f(x_0) < 0$, $f'(x_0) = 0$, and $f''(x_0) \neq 0$, it can be concluded that f has ...

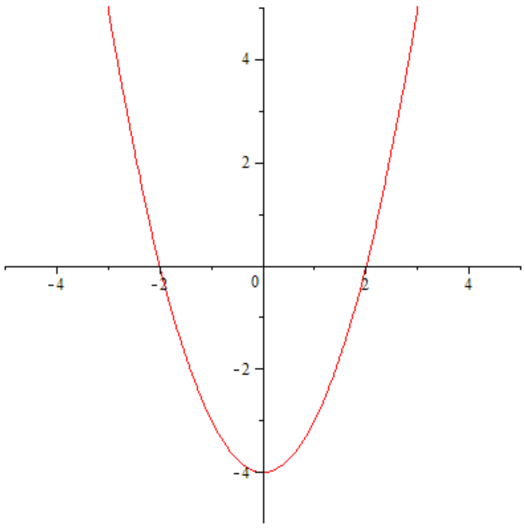
- ☐ ... no local minimum at x_0
- ☐ ... no local maximum at x_0
- ☐ ... no point of inflection at x_0
- ☐ ... a point of inflection at x_0

c) The global maximum of a function ...

- ☐ ... is always a local maximum.
- ☐ ... can be a local minimum.
- ☐ ... can be a local maximum.
- ☐ ... always exists.

Answers

13.1 a) $f(x) = x^2 - 4$

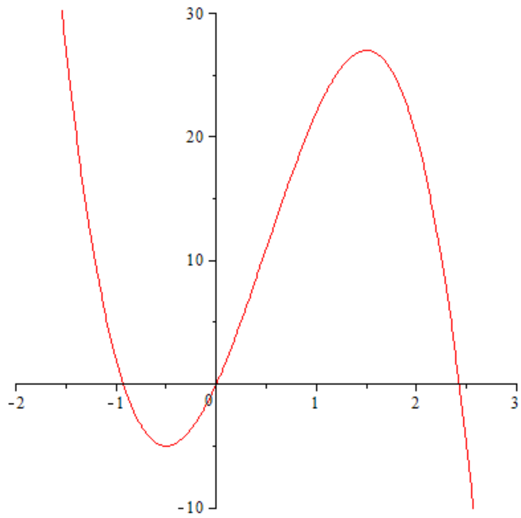


$f'(x) = 2x$
$f''(x) = 2$
$f'''(x) = 0$

i) $f'(x) = 0$ at $x_1 = 0$
$f''(x_1) = 2 > 0$ $\Rightarrow$ local minimum at $x_1 = 0$
no local maximum

ii) $f''(x) = 2 \neq 0$ for all x $\Rightarrow$ no point of inflection

b) $f(x) = -8x^3 + 12x^2 + 18x$

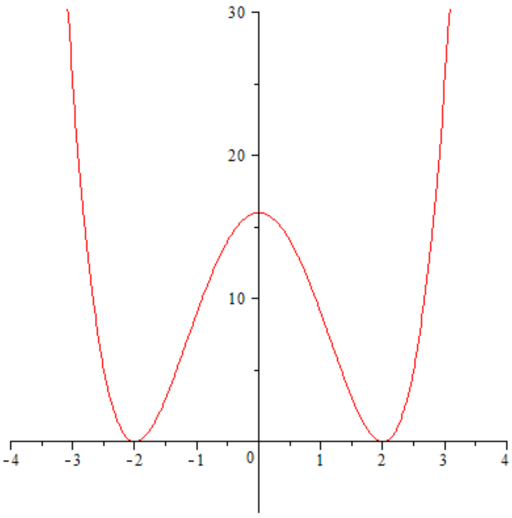


$f'(x) = -24x^2 + 24x + 18$
$f''(x) = -48x + 24$
$f'''(x) = -48$

i) $f'(x) = 0$ at $x_1 = -\frac{1}{2}$ and $x_2 = \frac{3}{2}$
$f''(x_1) = 48 > 0$ $\Rightarrow$ local minimum at $x_1 = -\frac{1}{2}$
$f''(x_2) = -48 < 0$ $\Rightarrow$ local maximum at $x_2 = \frac{3}{2}$

ii) $f''(x) = 0$ at $x_3 = \frac{1}{2}$
$f'''(x_3) = -48 \neq 0$ $\Rightarrow$ point of inflection at $x_3 = \frac{1}{2}$

c) $s(t) = t^4 - 8t^2 + 16$

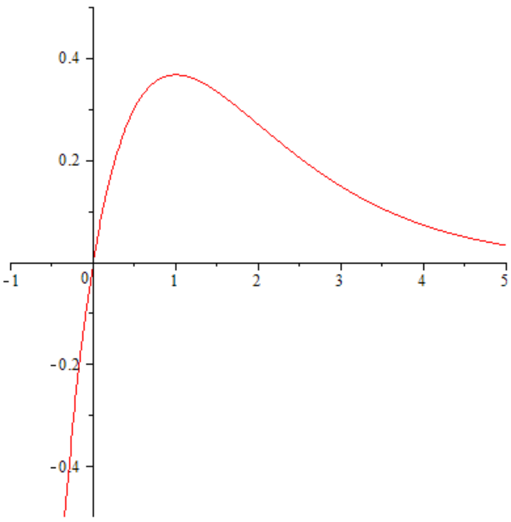


$s'(t) = 4t^3 - 16t$
$s''(t) = 12t^2 - 16$
$s'''(t) = 24t$

i) $s'(t) = 0$ at $t_1 = 0$, $t_2 = -2$, and $t_3 = 2$
$s''(t_1) = -16 < 0$ $\Rightarrow$ local maximum at $t_1 = 0$
$s''(t_2) = 32 > 0$ $\Rightarrow$ local minimum at $t_2 = -2$
$s''(t_3) = 32 > 0$ $\Rightarrow$ local minimum at $t_3 = 2$

ii) $s''(t) = 0$ at $t_4 = -\frac{2}{\sqrt{3}}$ and $t_5 = \frac{2}{\sqrt{3}}$
$s'''(t_4) = -\frac{48}{\sqrt{3}} \neq 0$ $\Rightarrow$ point of inflection at $t_4 = -\frac{2}{\sqrt{3}}$
$s'''(t_5) = \frac{48}{\sqrt{3}} \neq 0$ $\Rightarrow$ point of inflection at $t_5 = \frac{2}{\sqrt{3}}$

d) $f(x) = x\, e^{-x}$

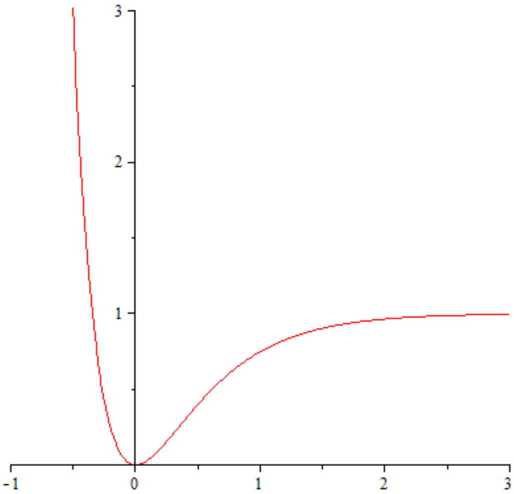


$f'(x) = e^{-x} - x\, e^{-x} = (1 - x)\, e^{-x}$
$f''(x) = -e^{-x} - (1 - x)\, e^{-x} = (x - 2)\, e^{-x}$
$f'''(x) = e^{-x} - (x - 2)\, e^{-x} = (3 - x)\, e^{-x}$

i) $f'(x) = 0$ at $x_1 = 1$
$f''(x_1) = -\frac{1}{e} < 0$ ⇒ local maximum at $x_1 = 1$
no local minimum

ii) $f''(x) = 0$ at $x_2 = 2$
$f'''(x_2) = \frac{1}{e^2} \neq 0$ ⇒ point of inflection at $x_2 = 2$

e) * $f(x) = (1 - e^{-2x})^2 = 1 - 2\,e^{-2x} + e^{-4x}$



$f'(x) = 4\,e^{-2x} - 4\,e^{-4x} = 4\,e^{-2x}\,(1 - e^{-2x})$
$f''(x) = -8\,e^{-2x} + 16\,e^{-4x} = 8\,e^{-2x}\,(2\,e^{-2x} - 1)$
$f'''(x) = 16\,e^{-2x} - 64\,e^{-4x} = 16\,e^{-2x}\,(1 - 4\,e^{-2x})$

i) $f'(x) = 0$ at $x_1 = 0$
$f''(x_1) = 8 > 0$ ⇒ local minimum at $x_1 = 0$
no local maximum

ii) $f''(x) = 0$ at $x_2 = \frac{\ln(2)}{2} = 0.34\ldots$
$f'''(x_2) = -8 \neq 0$ ⇒ point of inflection at $x_2 = 0.34\ldots$

f) * $V'(r) = -D\left(-\frac{2a}{r^2} + \frac{2a^2}{r^3}\right) = \frac{2aD}{r^2}\left(1 - \frac{a}{r}\right)$
$V''(r) = -D\left(\frac{4a}{r^3} - \frac{6a^2}{r^4}\right) = \frac{2aD}{r^3}\left(\frac{3a}{r} - 2\right)$
$V'''(r) = -D\left(-\frac{12a}{r^4} + \frac{24a^2}{r^5}\right) = \frac{12aD}{r^4}\left(1 - \frac{2a}{r}\right)$

i) $V'(r) = 0$ at $r_1 = a$
$V''(r_1) = \frac{2D}{a^2} > 0$ ⇒ local minimum at $r_1 = a$
no local maximum

ii) $V''(r) = 0$ at $r_2 = \frac{3a}{2}$
$V'''(r_2) = -\frac{64D}{81a^3} \neq 0$ ⇒ point of inflection at $r_2 = \frac{3a}{2}$

13.2 (Sole) **local** maximum at $x_1 = \frac{100}{3}$ → 33 or 34
P(33) = 51'843 CHF
P(34) = 51'816 CHF
$P(x) < P(x_1)$ if $x \neq x_1$ as there is no local minimum
⇒ P = 51'843 CHF is the **global** maximum profit at x = 33.

13.3 $\overline{C}(x) = \frac{C(x)}{x} = \left(\frac{1}{4}x + 4 + \frac{100}{x}\right) \cdot 100$ CHF
$\overline{C}(x)$ has a (sole) **local** minimum at $x_1 = 20$.
$\overline{C}(20) = 1400$ CHF
$\overline{C}(x) > \overline{C}(x_1)$ if $x \neq x_1$ as there is no local maximum.
$\Rightarrow$ $\overline{C} = 1400$ CHF is the **global** minimum average cost at $x = 20$.

13.4 $P(x)$ has a **local** maximum at $x_1 = 15$ and a **local** minimum at $x_2 = 20$.
$P(x_1) = 20'250$ CHF
$P(x) < P(x_1)$ if $x < x_1$ as there is no local minimum on the interval $x < x_1$.
$P(30) = 27'000$ CHF $> 20'250$ CHF (!)
$\Rightarrow$ $P = 27'000$ CHF is the **global** maximum profit at the endpoint $x = 30$.

13.5 $P(x)$ has a point of inflection at $x_1 = 10$.
$P(10) = 200 \cdot 1000$ CHF $= 200'000$ CHF
$\Rightarrow$ point of inflection (10 | 200’000 CHF), i.e. when $x = 10$ (in the year 2020) and $P = 200'000$ CHF

13.6 a) 4th statement

b) 3rd statement

c) 3rd statement