Exercises 8 Exponential function and equations
Compound interest, exponential function

Objectives

- be able to perform compound interest calculations.
- be able to graph an exponential function out of its equation.
- be able to determine the equation of an exponential function out of the coordinates of two points of the graph.
- be able to treat applied tasks by means of an exponential function.

Problems

Compound interest

8.1 Compound interest at an interest rate r is paid on an initial capital C_0.

a) Assume an initial capital C_0 = 1000.00 CHF, and an interest rate r = 2%.
Determine the capital after one, two, three, four, and five compounding periods.

b) Try to develop a formula which allows you to calculate the capital C_n after n compounding periods for any values of C_0, r, and n.

c) Solve the formula that you have developed in b) for C_0 and r.

8.2 8000 CHF are invested for 10 years at an annual interest rate of 12%, compounded annually.

Determine the capital after the 10 years.

8.3 A capital amounts to 10'000 CHF after being invested for 10 years at an annual interest rate of 6%, compounded annually.

Determine the initial capital.

8.4 10'000 CHF are invested at a certain annual interest rate, compounded annually, and amount to 14'000 CHF in 7 years.

Determine the annual interest rate.

8.5 Ms Smith wants to invest 150'000 CHF for five years. Bank A offers an annual interest rate of 6.5%, compounded annually. Bank B offers to pay 200'000 CHF after five years.

Decide which bank makes the better offer. Give reasons for your answer.

8.6 Mary Stahley invested 2500 CHF in a 36-month certificate of deposit (CD) that earned 8.5% annual **simple** interest. When the CD matured, she invested the full amount in a mutual fund that had an annual growth equivalent to 18%, **compounded** annually.

Determine the worth of the mutual fund 9 years later.

8.7 A capital is invested for 4 years at 4% and for 3 more years at 6%, compounded annually. Eventually, the capital amounts to 72'000 CHF.

a) Determine the initial capital.

b) Determine the average interest rate with respect to the whole period of time.

8.8 An unknown initial capital is invested at an unknown annual interest rate, compounded annually. After 2 years, the capital amounts to 5'891.74 CHF (rounded), and after another 5 years the capital is 6'997.54 CHF (rounded).

Determine both initial capital (rounded to 100 CHF) and annual interest rate (rounded to 0.1%).

8.9 A capital pays interest, compounded annually.

Determine the annual interest rate such that the capital doubles in 20 years.

8.10 3200 CHF is invested for 5 years at a nominal annual interest rate of 8%, compounded quarterly.

Determine the capital after the 5 years.

8.11 Parents want to deposit money in an account earning 10% (nominal annual interest rate), compounded monthly, so that it will grow to 40'000 CHF for their son's college tuition in 18 years.

Determine the amount of money the parents have to pay in.

8.12 A capital is invested at a nominal annual interest rate of 6%.

Determine by how many percent the capital grows in one year if interest is compounded ...

a) ... annually.

b) ... semiannually.

c) ... quarterly.

d) ... monthly.

e) ... daily (1 year = 360 days).

Exponential function

8.13 Look at the following exponential function:

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto y = f(x) = 2^x$$

a) Establish a table of values of f for the interval $-3 \leq x \leq 3$.

b) Draw the graph of f in the interval $-3 \leq x \leq 3$ into a Cartesian coordinate system.

8.14 Graph the following exponential functions into one coordinate system:

$$f_1: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto y = f_1(x) = 2^x$$

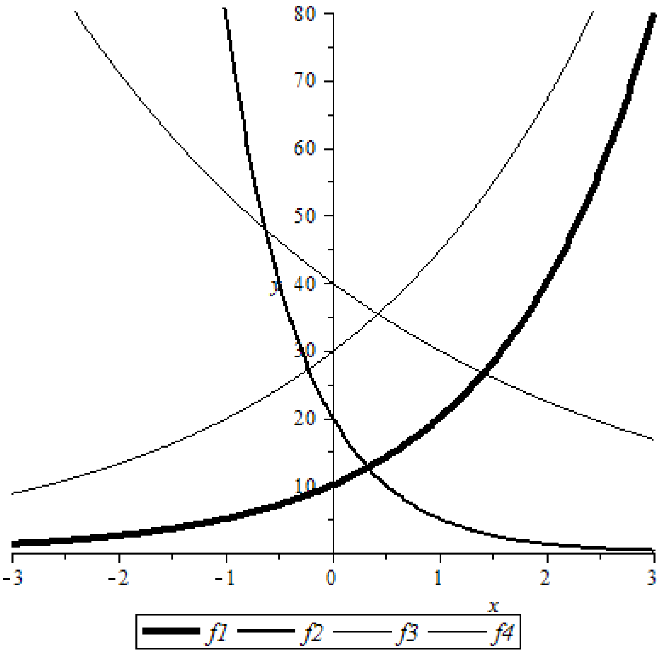
$$f_2: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto y = f_2(x) = 0.2^x$$

$$f_3: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto y = f_3(x) = 3 \cdot 0.5^x$$

$$f_4: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto y = f_4(x) = -2 \cdot 3^x$$

8.15 (see next page)

8.15 Look at the graphs of the exponential functions f_1, f_2, f_3, and f_4:



Determine the equations of the four functions, i.e. y = f(x) = ...

8.16 The graph of an exponential function contains the points P and Q.

Determine the equation of the exponential function.

a) P(1|12) Q(3|192)

b) P(0|1.02) Q(1|1.0302)

c) P(5|16) $Q\left(9|\frac{1}{16}\right)$

8.17 A flat that 20 years ago was worth 160'000 CHF has increased in value by 4% each year due to the market situation.

Determine what the flat is worth today.

8.18 A machine is valued at 10'000 CHF. The depreciation at the end of each year is 20% of its value at the beginning of the year.

Determine its value at the end of 4 years.

8.19 The size of a certain bacteria culture grows exponentially. At 8 a.m. and 11 a.m. the number of bacteria was 2'300 and 18'400, respectively.

Determine the number of bacteria at 1.30 p.m.

8.20 (see next page)

8.20 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) In a compound interest scheme ...

- ☐ ... the graph that represents the growth of the capital is a parabola.
- ☐ ... the interest paid at the end of each period only depends on the interest rate.
- ☐ ... the interest rate depends on the capital of the previous period.
- ☐ ... the capital grows exponentially.

b) The graph of an exponential function ...

- ☐ ... is a parabola.
- ☐ ... is a hyperbola.
- ☐ ... never intersects the y-axis.
- ☐ ... never touches the x-axis.

c) If a quantity grows exponentially in time ...

- ☐ ... the growth factor itself grows.
- ☐ ... the growth factor depends on the initial value.
- ☐ ... the quantity doubles in one year if the annual growth factor is 100%.
- ☐ ... the quantity doubles in constant time intervals.

Answers

8.1 a) $C_0 = 1000.00$ CHF $C_1 = 1020.00$ CHF $C_2 = 1040.40$ CHF

$C_3 = 1061.21$ CHF (rounded) $C_4 = 1082.43$ CHF (rounded) $C_5 = 1104.08$ CHF (rounded)

b) $C_n = C_0 (1 + r)^n$

c) see formulary

8.2 $C_n = C_0 (1 + r)^n$ where $C_0 = 8000$ CHF, $r = 12\%$, $n = 10$

$\Rightarrow$ $C_{10} = 24'846.79$ CHF (rounded)

8.3 $C_0 = \frac{C_n}{(1+r)^n}$ where $C_n = 10'000$ CHF, $r = 6\%$, $n = 10$

$\Rightarrow$ $C_0 = 5'583.95$ CHF (rounded)

8.4 $r = \sqrt[n]{\frac{C_n}{C_0}} - 1$ where $C_0 = 10'000$ CHF, $C_n = 14'000$ CHF, $n = 7$

$\Rightarrow$ $r = 4.9\%$ (rounded)

8.5 Bank A: $C_5 = 205'513.00$ CHF (rounded)
Bank B: $C_5 = 200'000.00$ CHF

8.6 13'916.24 CHF

2 periods: 3 years of simple interest, 9 years of compound interest

- 3 years of simple interest:

$C_n = C_0(1 + nr)$ where $C_0 = 2500$ CHF, $r = 8.5\%$, $n = 3$

$\Rightarrow$ $C_3 = 3137.50$ CHF

- 9 years of compound interest:

$C_n = C_0 (1 + r)^n$ where $C_0 = \ldots$ (= C_3 after first 3 years), $r = 18\%$, $n = 9$

$\Rightarrow$ $C_9 = 13'916.24$ CHF (rounded)

8.7 a) $C_0 = 51'675$ CHF (rounded)

Hints:
- First, look at the second period (3 years, starting after 4 years from now), and calculate the capital at the beginning of this second period.
- Then, calculate the initial capital.

b) $i = 4.85\%$ (gerundet)

Hints:
- There are two possible solution processes:
- **Process 1**
  The average interest rate r must be such that:
  $C_n = C_0 (1 + r)^n = C_0 (1 + r_1)^{n_1} (1 + r_2)^{n_2}$ and $n_1 + n_2 = n$
  where r_1 = interest rate in the first n_1 periods, r_2 = interest rate in the remaining n_2 periods
- **Process 2**
  The average interest rate r must be such that:
  $C_n = C_0 (1 + r)^n$
  where C_0 = initial capital, C_n = capital after the whole n periods

8.8 $r = 3.5\%$, $C_0 = 5'500.00$ CHF

Hints:
- First, look at the second period of 5 years, where $C_0 = 5'891.74$ CHF and $C_5 = 6'997.54$ CHF.
- The 5'891.74 CHF can be considered as the capital C_2 at the end of the first 2 years if C_0 is the initial capital at the beginning of the whole 7 years.

8.9 $r = \sqrt[n]{\frac{C_n}{C_0}} - 1$ where $n = 20$

$C_n = 2 \cdot C_0$

$\Rightarrow \; r = \sqrt[20]{2} - 1 = 3.5\%$ (rounded)

8.10 $C_n = C_0 (1 + r)^n$ where $C_0 = 3200$ CHF, $r = \frac{8\%}{4} = 2\%$, $n = 5 \cdot 4 = 20$

$\Rightarrow \; C_{20} = 4755.03$ CHF (rounded)

8.11 $C_0 = \frac{C_n}{(1+r)^n}$ where $C_n = 40'000$ CHF, $r = \frac{10\%}{12}$, $n = 18 \cdot 12 = 216$

$\Rightarrow \; C_0 = 6661.46$ CHF (rounded)

8.12 Capital after 1 year

$C_n = C_0 (1 + r)^n$ (n = number of compounding periods per year, r = interest rate per compounding period)

$C_n = C_0 (1 + x)$ (x = asked percentage)

$\Rightarrow \; x = (1 + r)^n - 1$

a) $n = 1$, $r = 6\%$ $x = 6\%$

b) $n = 2$, $r = \frac{6\%}{2}$ $x = 6.09\%$

c) $n = 4$, $r = \frac{6\%}{4}$ $x = 6.136\%$ (rounded)

d) $n = 12$, $r = \frac{6\%}{12}$ $x = 6.168\%$ (rounded)

e) $n = 360$, $r = \frac{6\%}{360}$ $x = 6.183\%$ (rounded)

8.13 ...

8.14 ...

8.15 $y = f_1(x) = 10 \cdot 2^x$ ($c = 10$, $a = 2$)

$y = f_2(x) = 20 \cdot 0.25^x$ ($c = 20$, $a = 0.25$)

$y = f_3(x) = 30 \cdot 1.5^x$ ($c = 30$, $a = 1.5$)

$y = f_4(x) = 40 \cdot 0.75^x$ ($c = 40$, $a = 0.75$)

8.16 (see next page)

8.16 a) $y = f(x) = 3 \cdot 4^x$

Hints:
- The equation of an exponential function is $y = f(x) = c \cdot a^x$
- If P(1|12) and Q(3|192) are points of the graph of the exponential function, their coordinates must fulfil the equation of the exponential function, i.e. $12 = f(1) = c \cdot a^1$ and $192 = f(3) = c \cdot a^3$
- Solve the two equations for c and a.

b) $y = f(x) = 1.02 \cdot 1.01^x$

c) $y = f(x) = 16'384 \cdot 0.25^x$

8.17 350'580 CHF (rounded)

Hint:
- The relation between time t (t = number of years elapsed since 20 years ago) and the value V of the house is an exponential function:
$V = f(t) = V_0 \cdot a^t$
where V = value at time t, V_0 = initial value (at t = 0) = 160'000 CHF, a = growth factor = 1 + 4% = 1.04

8.18 4'096 CHF

8.19 104'086 (rounded)

8.20 a) 4th statement

b) 4th statement

c) 4th statement