

Exercises 3 Linear function and equations

Linear function, simple interest, cost, revenue, profit, break-even

Objectives

- be able to think of a relation between two quantities as a function.
- be able to determine the domain, the codomain, the range of a given function.
- be able to draw the graph of a given linear function.
- be able to determine slope and intercept of a linear function.
- know some examples of linear functions in economic and everyday life applications.
- know and understand what simple interest is.
- be able to perform simple interest calculation.
- know and understand what fixed costs, variable costs, total costs, total revenue, total profit, and break-even value are.
- be able to apply the concept of linear functions to a new problem.

Problems

- 3.1 A taxi driver charges the following fare:

8.00 CHF plus 1.50 CHF per kilometre

Think of the taxi fare as a function f .

- Determine the domain D , the codomain C , and the range R of the function.
- Draw the graph of the function f .

- 3.2 The taxi fare as described in problem 3.1 can be thought of as a linear function which assigns a fare to each distance:

$$\begin{aligned} f: \mathbb{R}^+ &\rightarrow \mathbb{R}^+ \\ x &\mapsto y = f(x) = ax + b \end{aligned}$$

where: $x = \text{distance/km}$
 $y = \text{fare/CHF}$

Determine the values of a and b .

- 3.3 Find at least two more examples of linear functions in economics or in an everyday life context.

- 3.4 State both slope and intercept of the linear functions below, and draw the graphs of the functions:

- $f: \mathbb{R} \rightarrow \mathbb{R}$
 $x \mapsto y = f(x) = -2$
- $f: \mathbb{R} \rightarrow \mathbb{R}$
 $x \mapsto y = f(x) = 2x - 6$
- $f: \mathbb{R} \rightarrow \mathbb{R}$
 $x \mapsto y = f(x) = -x + 3$

- 3.5 Simple interest at an annual rate of 0.5% is paid on an initial bank balance of 5000 CHF.

- Determine the interest that is paid each year.
- Determine the balance after ten years' time.
- Determine both slope and intercept of the corresponding linear function.

- 3.6 In general, if an initial capital C_0 pays simple interest at an annual rate r (e.g. $r = 1.5\% = 0.015$), the capital C_n after n years is given by the formula below (see [formulary](#)):

$$C_n = C_0 (1 + nr)$$

- a) Verify that the given formula is correct.
- b) Determine both slope and intercept of the corresponding linear function.

- 3.7 An initial capital $C_0 = 1200$ CHF pays simple interest at an annual interest rate of 1.5%.

- a) Determine after how many years the capital will exceed 2000 CHF.
- b) Determine at what annual interest rate (rounded to 0.05%) the capital will exceed 2000 CHF after 20 years' time.

Hint:

- Use the formula given in problem 3.6 and solve it for n and r respectively.

- 3.8 A satellite phone company offers three different tariffs:

Tariff A:	monthly basic fee of 10 CHF plus 0.20 CHF per minute
Tariff B:	monthly basic fee of 25 CHF plus 0.10 CHF per minute
Tariff C:	no basic fee, 0.60 CHF per minute

Think of the the three tariffs as linear functions.

- a) Draw the graphs of the three functions in one common coordinate system.
- b) Determine the total fee for each tariff for a monthly phone call duration of 1 hour.
- c) Determine the monthly phone call duration where tariff A is cheaper than tariff C.
- d) Determine the monthly phone call duration where tariff B is cheaper than tariff A.

- 3.9 (from: Bittinger, Ellenbogen: Calculus and its applications, Pearson 2007, ISBN 0-321-48543-2)

EXAMPLE 9 Business: Total Cost. Raggs, Ltd., a clothing firm, has **fixed costs** of \$10,000 per year. These costs, such as rent, maintenance, and so on, must be paid no matter how much the company produces. To produce x units of a certain kind of suit, it costs \$20 per suit (unit) in addition to the fixed costs. That is, the **variable costs** for producing x of these suits are $20x$ dollars. These costs are due to the amount produced and stem from items such as material, wages, fuel, and so on. The **total cost** $C(x)$ of producing x suits in a year is given by a function C :

$$C(x) = (\text{Variable costs}) + (\text{Fixed costs}) = 20x + 10,000.$$

- a) Graph the variable-cost, the fixed-cost, and the total-cost functions.
- b) What is the total cost of producing 100 suits? 400 suits?

- 3.10 (see next page)

3.10 (from: Bittinger, Ellenbogen: Calculus and its applications, Pearson 2007, ISBN 0-321-48543-2)

EXAMPLE 10 Business: Profit-and-Loss Analysis. When a business sells an item, it receives the *price* paid by the consumer (this is normally greater than the *cost* to the business of producing the item).

- a) The **total revenue** that a business receives is the product of the number of items sold and the price paid per item. Thus, if Raggs, Ltd., sells x suits at \$80 per suit, the total revenue $R(x)$, in dollars, is given by

$$R(x) = \text{Unit price} \cdot \text{Quantity sold} = 80x.$$

If $C(x) = 20x + 10,000$ (see Example 9), graph R and C using the same set of axes.

- b) The **total profit** that a business receives is the amount left after all costs have been subtracted from the total revenue. Thus, if $P(x)$ represents the total profit when x items are produced and sold, we have

$$P(x) = (\text{Total revenue}) - (\text{Total costs}) = R(x) - C(x).$$

Determine $P(x)$ and draw its graph using the same set of axes as was used for the graph in part (a).

- c) The company will *break even* at that value of x for which $P(x) = 0$ (that is, no profit and no loss). This is the point at which $R(x) = C(x)$. Find the **break-even value** of x .

3.11 Decide which statements are true or false. Put a mark into the corresponding box. In each problem a) to c), exactly one statement is true.

- a) ☐ Each straight line in a coordinate system can be considered as the graph of a linear function.
☐ The graph of each linear function is a straight line.
☐ If y is proportional to x , x is not necessarily proportional to y .
☐ The range of each linear function is $\mathbb{R}$.
- b) f cannot be a linear function if ...
☐ ... the graph of f is a straight line.
☐ ... $f(x) \neq x$ for at least one element x of the domain of f .
☐ ... the domain of f does not consist of all real numbers.
☐ ... $f(x) = ax + b$ and a depends on x .
- c) In a simple interest scheme ...
☐ ... the relation between time and capital does not correspond to a linear function.
☐ ... the interest paid at the end of each period depends on the capital at the end of the previous period.
☐ ... the interest paid at the end of each period is always the same amount of money.
☐ ... the capital doubles in less than 5 years if the annual interest rate is 20%.

Answers

- 3.1 a) $D = \mathbb{R}^+$ (distance/km)
 $C = \mathbb{R}^+$ (fare/CHF)
 $R = \{y: y \in \mathbb{R}^+ \text{ and } y > 8\}$
b) ...

3.2 $a = 1.5, b = 8$

3.3 ...

- 3.4 a) Slope $a = 0$, intercept $b = -2$
b) Slope $a = 2$, intercept $b = -6$
c) Slope $a = -1$, intercept $b = 3$

- 3.5 a) 25 CHF
b) 5250 CHF
c) $f: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$
 $x \mapsto y = f(x) = ax + b$
where: x = number of years from the beginning
 y = capital/CHF after x years
Slope $a = 25$, intercept $b = 5000$

- 3.6 a) Interest paid each year $= r \cdot C_0$
Capital C_n after n years $= C_0 + n \cdot (r \cdot C_0) = C_0 (1 + nr)$
b) Slope $a = r \cdot C_0$, intercept $b = C_0$
Hints:
- Compare the formula $C_n = C_0 (1 + nr)$ with the general form of the equation of a linear function.
- $C_n = C_0 (1 + nr) = an + b = f(n)$
- f with $f(x) = ax + b$ is a linear function.

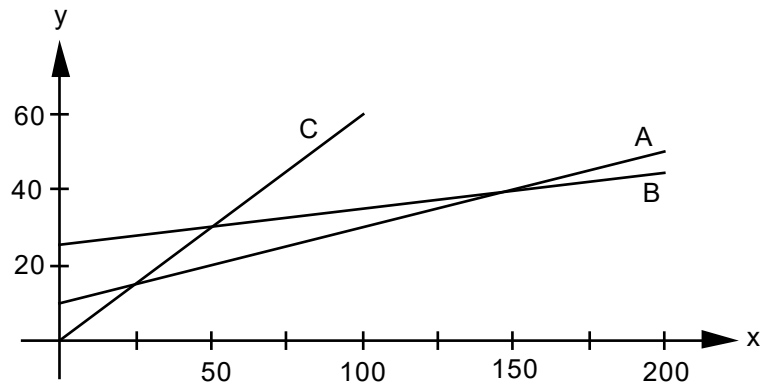
- 3.7 a) $n = \frac{\frac{C_n}{C_0} - 1}{r}$ where $C_0 = 1200$ CHF, $C_n = 2000$ CHF, $r = 1.5\% = 0.015$
 $\Rightarrow n = 44.4... \rightarrow 45$ years
b) $r = \frac{\frac{C_n}{C_0} - 1}{n}$ where $C_0 = 1200$ CHF, $C_n = 2000$ CHF
 $n = 20 \Rightarrow r = 0.03333... = 3.333...\%$
 $n = 19 \Rightarrow r = 0.03508... = 3.508...\%$
 $\Rightarrow r \in \{3.35\%, 3.40\%, 3.45\%, 3.50\%\}$

- 3.8 a) x = phone call duration/min
 y = fee/CHF
Tariff A: $A: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$
 $x \mapsto y = A(x) = 0.2x + 10$
Tariff B: (see next page)

Tariff B: $B: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$
 $x \mapsto y = B(x) = 0.1x + 25$

Tariff C: $C: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$
 $x \mapsto y = C(x) = 0.6x$

Direct proportionality: The fee is directly proportional to the phone call duration.



- b) Tariff A: 22 CHF
 Tariff B: 31 CHF
 Tariff C: 36 CHF

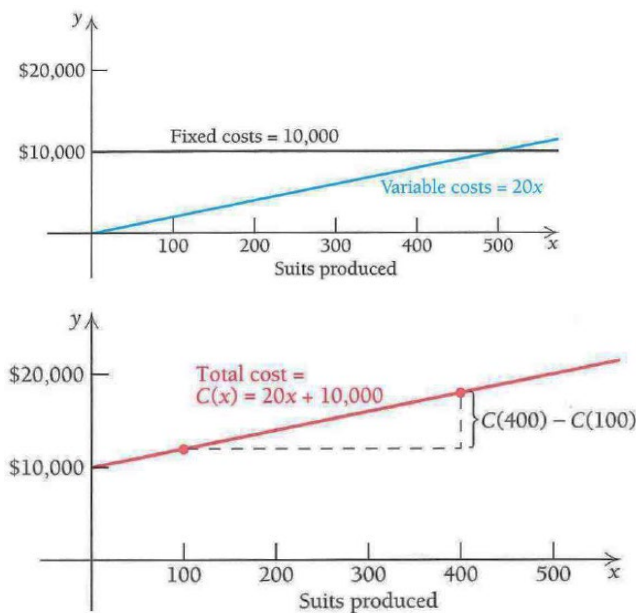
- c) over 25 min

Hint:

- Solve the equation $A(x) = C(x)$ for x .

- d) over 150 min

3.9 a)

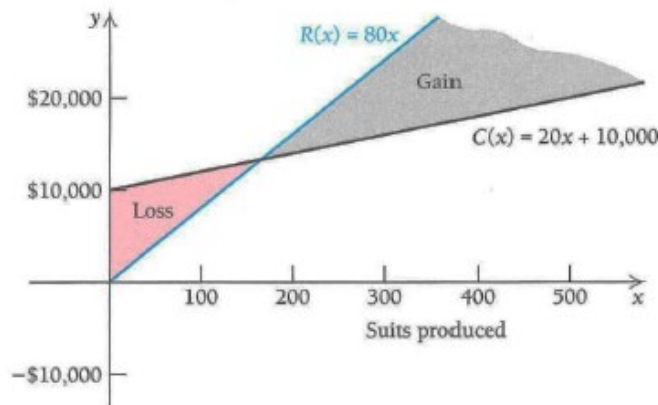


- b) $C(100) = (20 \cdot 100 + 10'000) = \$12'000$
 $C(400) = (20 \cdot 400 + 10'000) = \$18'000$

3.10 (see next page)

3.10

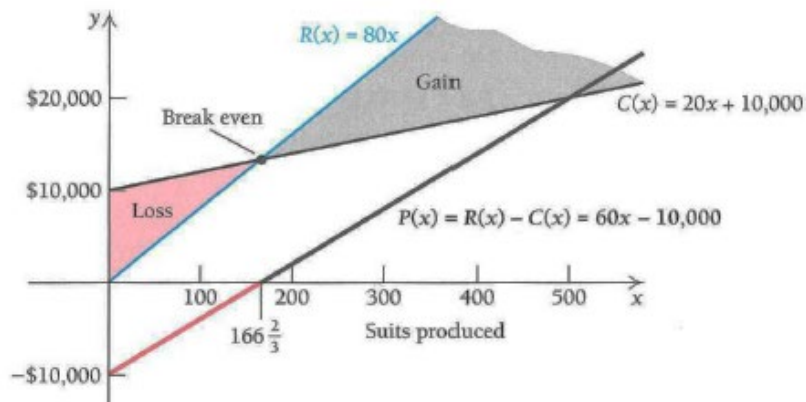
- a) The graphs of $R(x) = 80x$ and $C(x) = 20x + 10,000$ are shown below. When $C(x)$ is above $R(x)$, a loss will occur. This is shown by the region shaded red. When $R(x)$ is above $C(x)$, a gain will occur. This is shown by the region shaded gray.



- b) To find P , the profit function, we have

$$\begin{aligned} P(x) &= R(x) - C(x) = 80x - (20x + 10,000) \\ &= 60x - 10,000. \end{aligned}$$

The graph of $P(x)$ is shown by the heavy line. The red portion of the line shows a “negative” profit, or loss. The black portion of the heavy line shows a “positive” profit, or gain.



- c) To find the break-even value, we solve $R(x) = C(x)$:

$$\begin{aligned} R(x) &= C(x) \\ 80x &= 20x + 10,000 \\ 60x &= 10,000 \\ x &= 166\frac{2}{3}. \end{aligned}$$

How do we interpret the fractional answer, since it is not possible to produce $\frac{2}{3}$ of a suit? We simply round to 167. Estimates of break-even values are usually sufficient since companies want to operate well away from break-even values in order to maximize profit. ♦

- 3.11 a) 2nd statement
b) 4th statement
c) 3rd statement