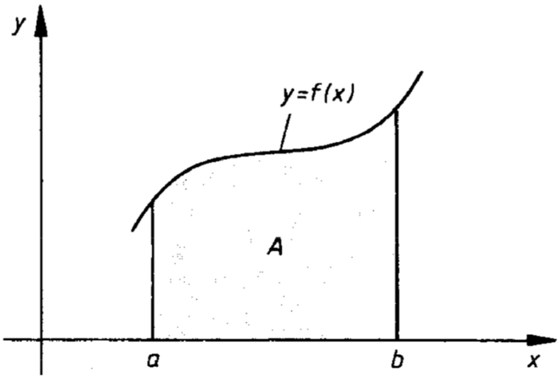
Definite integral

Area under a curve

$f: D \to \mathbb{R} \qquad (D \subseteq \mathbb{R})$
$\quad x \mapsto y = f(x)$

Suppose that $f(x) \geq 0$ on the interval $a \leq x \leq b$:

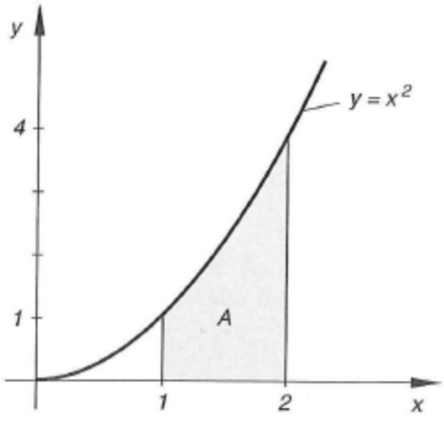


A = area between the graph of f and the x-axis on the interval $a \leq x \leq b$

Definition

The area A between the graph of f and the x-axis on the interval $a \leq x \leq b$ is the **definite integral** of f from a to b, denoted $\int_a^b f(x)\,dx$.
$A = \int_a^b f(x)\,dx$

Ex.: $f(x) = x^2$



$A = \int_1^2 x^2\,dx$

Fundamental theorem of calculus

$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ where F is any antiderivative of f

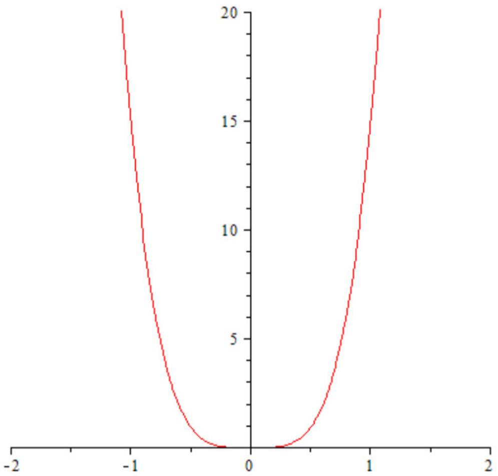
Ex.: 1. $f(x) = x^2$, $a = 1$, $b = 2$

$$\int_1^2 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_1^2 = \tfrac{1}{3}2^3 - \tfrac{1}{3}1^3 = \tfrac{8}{3} - \tfrac{1}{3} = \tfrac{7}{3} = 2.\overline{3}$$

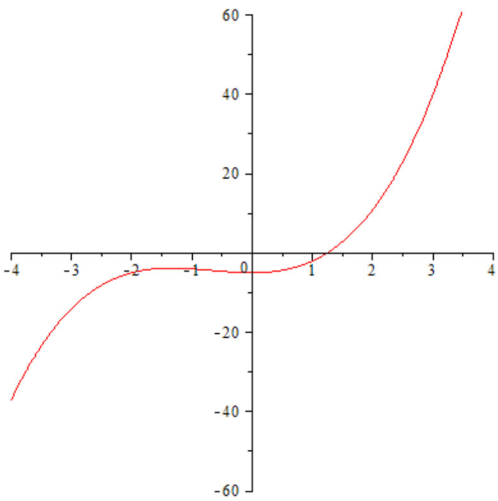
or: $$\int_1^2 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_1^2 = \tfrac{1}{3}[x^3]_1^2 = \tfrac{1}{3}\left(2^3 - 1^3\right) = \tfrac{7}{3} = 2.\overline{3}$$

2. $$\int_0^2 x^3\,dx = \left[\tfrac{1}{4}x^4\right]_0^2 = \tfrac{1}{4}[x^4]_0^2 = \tfrac{1}{4}\left(2^4 - 0^4\right) = 4$$

3. $$\int_{-1}^1 15x^4\,dx = \left[15\cdot\tfrac{1}{5}x^5\right]_{-1}^1 = 3[x^5]_{-1}^1 = 3\left(1^5 - (-1)^5\right) = 6$$



4. $$\int_2^3 (x^3 + 2x^2 - 5)\,dx = \left[\tfrac{1}{4}x^4 + 2\cdot\tfrac{1}{3}x^3 - 5x\right]_2^3 = \left(\tfrac{3^4}{4} + \tfrac{2\cdot 3^3}{3} - 5\cdot 3\right) - \left(\tfrac{2^4}{4} + \tfrac{2\cdot 2^3}{3} - 5\cdot 2\right) = \tfrac{287}{12} = 23.91\overline{6}$$



Consumer's Surplus

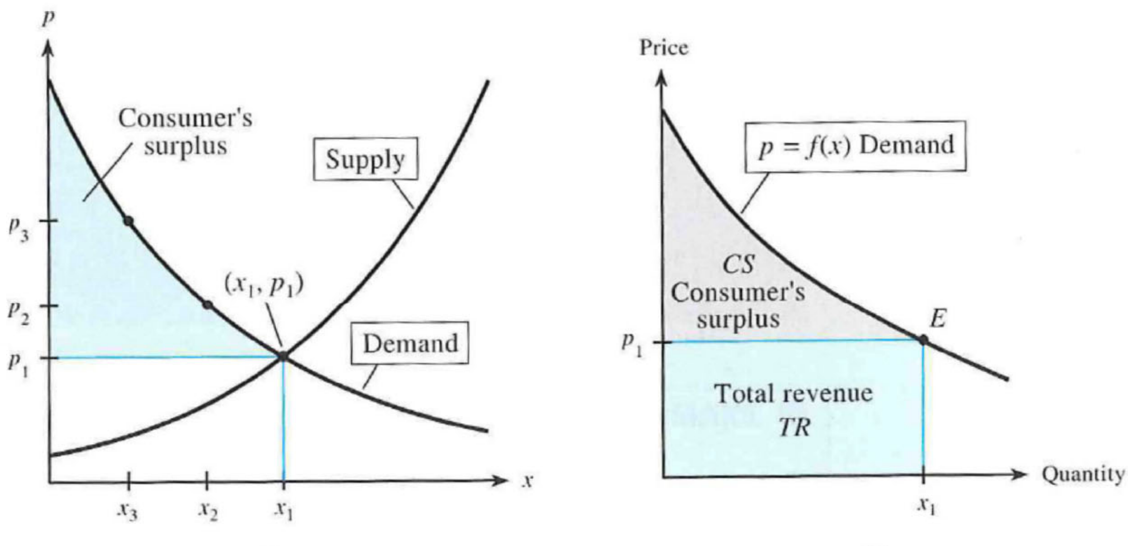
Suppose that the demand for a product is given by $p = f(x)$ and that the supply of the product is described by $p = g(x)$. The price p_1 where the graphs of these functions intersect is the **equilibrium price** (see Figure 13.21(a)). As the demand curve shows, some consumers (but not all) would be willing to pay more than p_1 for the product.

For example, some consumers would be willing to buy x_3 units if the price were p_3. Those consumers willing to pay more than p_1 are benefiting from the lower price. The total gain for all those consumers willing to pay more than p_1 is called the **consumer's surplus,** and under proper assumptions the area of the shaded region in Figure 13.21(a) represents this consumer's surplus.

Looking at Figure 13.21(b), we see that if the demand curve has equation $p = f(x)$, the consumer's surplus is given by the area between $f(x)$ and the x-axis from 0 to x_1, *minus* the area of the rectangle denoted TR:

$$CS = \int_0^{x_1} f(x)\,dx - p_1 x_1$$

Note that with equilibrium price p_1 and equilibrium quantity x_1, the product $p_1 x_1$ is the area of the rectangle that represents the total dollars spent by consumers and received as revenue by producers (see Figure 13.21(b)).



(a) (b)

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p. 904

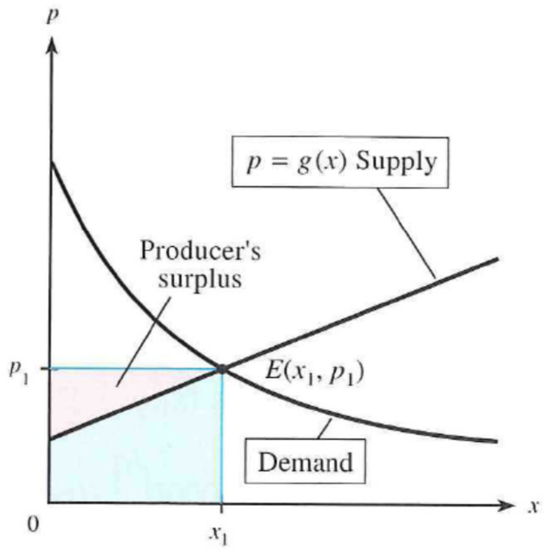
Producer's Surplus

When a product is sold at the equilibrium price, some producers will also benefit, for they would have sold the product at a lower price. The area between the line $p = p_1$ and the supply curve (from $x = 0$ to $x = x_1$) gives the producer's surplus (see Figure 13.23).

If the supply function is $p = g(x)$, the **producer's surplus** is given by the area between the graph of $p = g(x)$ and the x-axis from 0 to x_1 *subtracted from* the area of the rectangle $0x_1Ep_1$.

$$PS = p_1x_1 - \int_0^{x_1} g(x)\,dx$$

Note that p_1x_1 represents the total revenue at the equilibrium point.



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