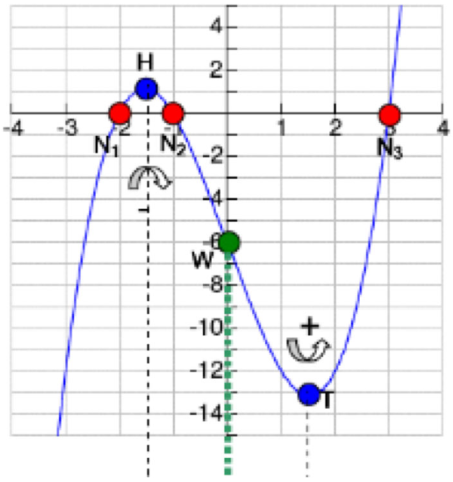
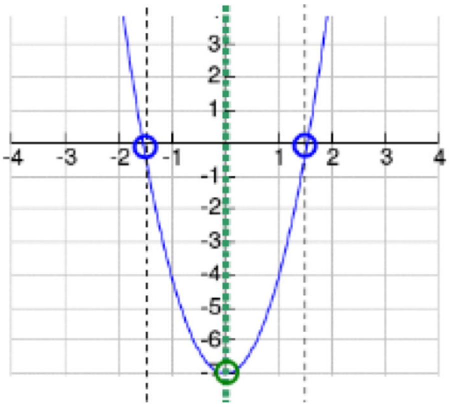
Increasing/decreasing, concavity

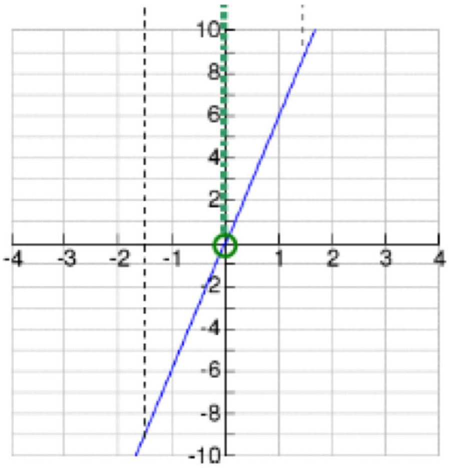
Ex.: $f(x) = x^3 - 7x - 6$



$f'(x) = 3x^2 - 7$



$f''(x) = 6x$



Increasing/decreasing

If the **first derivative** of a function f is **positive** at $x = x_0$, i.e. $f'(x_0) > 0$, f is **increasing** at $x = x_0$.

If the **first derivative** of a function f is **negative** at $x = x_0$, i.e. $f'(x_0) < 0$, f is **decreasing** at $x = x_0$.

Note: The reverse is also true:

If a function f is increasing at $x = x_0$, the first derivative of f at $x = x_0$ is positive, i.e. $f'(x_0) > 0$.

If a function f is decreasing at $x = x_0$, the first derivative of f at $x = x_0$ is negative, i.e. $f'(x_0) < 0$.

Concavity

If the **second derivative** of a function f is **positive** at $x = x_0$, i.e. $f''(x_0) > 0$, the graph of f is **concave up** (“left-hand bend”) at $x = x_0$.

If the **second derivative** of a function f is **negative** at $x = x_0$, i.e. $f''(x_0) < 0$, the graph of f is **concave down** (“right-hand bend”) at $x = x_0$.

Note: Here, the reverse is **not** true:

If the graph of a function f is concave up at $x = x_0$ (“left-hand bend”), the second derivative of f is not necessarily positive, but can be positive or equal to zero, i.e. $f''(x_0) > 0$ or $f''(x_0) = 0$.

If the graph of a function f is concave down at $x = x_0$ (“right-hand bend”), the second derivative of f is not necessarily negative, but can be negative or equal to zero, i.e. $f''(x_0) < 0$ or $f''(x_0) = 0$.

Local maxima/minima

A function f has a **local maximum** at $x = x_0$ if the tangent to the graph of f at $x = x_0$ is horizontal and if the graph of f is concave down (“right-hand bend”) at $x = x_0$.

This applies if $f'(x_0) = 0$ (necessary condition) and $f''(x_0) < 0$ (sufficient condition).

A function f has a **local minimum** at $x = x_0$ if the tangent to the graph of f at $x = x_0$ is horizontal and if the graph of f is concave up (“left-hand bend”) at $x = x_0$.

This applies if $f'(x_0) = 0$ (necessary condition) and $f''(x_0) > 0$ (sufficient condition).

Global maximum/minimum

The **global maximum/minimum** of a continuous function f is either a local maximum/minimum of f or the value of f at one of the endpoints of the domain.

Points of inflection

A function f has a **point of inflection** at $x = x_0$ if the graph of f changes its concavity from concave up to concave down (or vice versa) at $x = x_0$.

This applies if $f''(x_0) = 0$ (necessary condition) and $f'''(x_0) \neq 0$ (sufficient condition).

Ex.: (see next page)

Ex.: $f(x) = x^3 - 7x - 6$ (see page 1) $\Rightarrow f'(x) = 3x^2 - 7$

$\Rightarrow f''(x) = 6x$

$\Rightarrow f'''(x) = 6$

Local maxima/minima

$f'(x) = 0$ at $x_1 = \sqrt{\frac{7}{3}} = 1.52\ldots$ and $x_2 = -\sqrt{\frac{7}{3}} = -1.52\ldots$

$f''(x_1) = 6 \cdot \sqrt{\frac{7}{3}} = 9.16\ldots > 0$ $\Rightarrow$ local minimum at $x_1 = \sqrt{\frac{7}{3}}$

$f''(x_2) = -6 \cdot \sqrt{\frac{7}{3}} = -9.16\ldots < 0$ $\Rightarrow$ local maximum at $x_2 = -\sqrt{\frac{7}{3}}$

Global maximum/minimum

Ex.: $D = \{x: x \in \mathbb{R} \text{ and } 0 \leq x \leq 4\}$ $\Rightarrow$ global maximum at $x = 4$ (endpoint of domain)
$\Rightarrow$ global minimum at $x = x_1 = \sqrt{\frac{7}{3}}$ (local minimum)

Ex.: $D = \{x: x \in \mathbb{R} \text{ and } -4 \leq x \leq 3\}$ $\Rightarrow$ global maximum at $x = x_2 = -\sqrt{\frac{7}{3}}$ (local maximum)
$\Rightarrow$ global minimum at $x = -4$ (endpoint of domain)

Points of inflection

$f''(x) = 0$ at $x_3 = 0$

$f'''(x_3) = 6 \neq 0$ $\Rightarrow$ point of inflection at $x_3 = 0$

Financial mathematics

Marginal cost / Marginal revenue / Marginal profit function
= first derivative of the cost/revenue/profit function

Ex.: Cost function	$C(x) = (2x^2 + 120)$ CHF
$\Rightarrow$ Marginal cost function	$C'(x) = 4x$ CHF
Revenue function	$R(x) = (-x^2 + 168x)$ CHF
$\Rightarrow$ Marginal revenue function	$R'(x) = (-2x + 168)$ CHF
Profit function	$P(x) = R(x) - C(x) = (-3x^2 + 168x - 120)$ CHF
$\Rightarrow$ Marginal profit function	$P'(x) = (-6x + 168)$ CHF

Average cost / Average revenue / Average profit function

Average cost function / Unit cost function $\overline{C}(x) := \frac{C(x)}{x}$ where $C(x)$ = cost function

Ex.: Cost function $C(x) = (3x^2 + 4x + 2)$ CHF
$\Rightarrow$ Average cost function $\overline{C}(x) = \left(3x + 4 + \frac{2}{x}\right)$ CHF

Average revenue function $\overline{R}(x) := \frac{R(x)}{x}$ where $R(x)$ = revenue function

Average profit function $\overline{P}(x) := \frac{P(x)}{x}$ where $P(x)$ = profit function