Exercises 12 — Differentiation rules
Coefficient, sum, product, exponential function, higher-order derivatives

Objectives

- be able to apply the coefficient, sum, and product rules to determine the derivative of a function.
- be able to determine a higher-order derivative of a function.

Problems

12.1 Determine the derivative by applying the **coefficient rule**:

a) $f(x) = 3x^5$ b) $f(x) = -4x^3$ c) $f(x) = -x^{10}$

d) $f(x) = a \cdot x^3$ e) $f(x) = n \cdot x^{n-1}$ f) $f(x) = 9 \cdot 3^x$

g) $s(t) = \frac{1}{2} g \cdot t^2$ h) $S(T) = \alpha \cdot T^4$ i) $C(x) = (-3x)^3$

12.2 Determine the derivative by applying the **sum rule**:

a) $f(x) = x^5 + x^6$ b) $f(x) = x^{10} - x^9$ c) $f(x) = 1 + x + 3x^3$

d) $f(x) = \frac{1}{4} x^4 + 3x^2 - 2$ e) $f(x) = 3x^2(x - 2)$ f) $f(x) = - 3x^8 + x^5 - 3x + 99$

g) $f(x) = ax^2 + bx + c$ h) $f(x) = 3(a^2 - 2ax + x^2)$ i) $f(x) = \frac{x^3}{3} - \frac{3}{x^3}$

j) $s(t) = s_0 + v_0 t + \frac{1}{2} g \cdot t^2$ k) $V(r) = - \frac{a}{r} + \frac{b}{r^2}$ l) $C(n) = C_0(1 + nr)$

Hint:
- In some problems, the coefficient rule is needed, too.

12.3 Determine the derivative by applying the **product rule**:

a) $f(x) = x \cdot e^x$ b) $f(x) = x^3 \cdot 3^x$

c) $f(x) = - 2x^5(x - 1)$ d) $f(x) = (2x - 1) \cdot e^x$

e) $f(x) = (2x - 1)(- 3x^2 - x + 1)$ f) $V(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3}\right)$

Hint:
- In some problems, the coefficient and/or the sum rule(s) is/are needed, too.

12.4 Determine the derivative of the exponential functions below:

a) $f(x) = e^{4x}$ b) $f(x) = e^{-x}$

c) $f(x) = e^{-x^2}$ d) $f(x) = e^{x^2-2x+5}$

12.5 Determine the derivative of the functions below. Apply the appropriate differentiation rule(s). Simplify and factorise the derivative as far as possible:

a) $f(x) = (x - 2)\, e^{2x}$ b) $f(x) = (2 - x^2)\, e^{-x}$

c) $f(x) = (3x^3 - 2x^2 + x - 1)\, e^{-2x}$ d) $P(v) = av^2\, e^{-bv^2}$

12.6 (see next page)

12.6 Determine the derivatives (rates of change) below:

a) $f'(2)$ with function f in 12.1 b)

b) $s'(4)$ with function s in 12.1 g)

c) $f'(-1)$ with function f in 12.2 g)

d) $P'(1)$ with function P in 12.5 d)

12.7 Determine the second and third derivatives of the functions below. Simplify and factorise the higher-order derivatives as far as possible:

a) Function f in 12.1 a)

b) Function f in 12.2 g)

c) Function f in 12.3 a)

d) Function f in 12.4 c)

Hint:
- You have already determinded the first derivatives of the corresponding functions.

12.8 Determine the indicated higher-order derivatives:

a) $f''(-1)$ with function f in 12.1 a)

Hint:
- You have already determined $f''(x)$ in 12.7 a).

b) $f'''(2)$ with function f in 12.4 c)

Hint:
- You have already determined $f'''(x)$ in 12.7 d).

12.9 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) The third derivative of a function is a ...

- ☐ ... constant function if the second derivative is a quadratic function.
- ☐ ... quadratic function if the second derivative is a linear function.
- ☐ ... linear function if the first derivative is a quadratic function.
- ☐ ... constant function if the first derivative is a quadratic function.

b) The derivative of a ...

- ☐ ... product is the product of the derivatives of the single factors.
- ☐ ... product is the sum of the derivatives of the single factors.
- ☐ ... sum is the sum of the derivatives of the single addends.
- ☐ ... constant is the constant itself.

c) If $f(x) = c \cdot g(x) \cdot h(x)$ then $f'(x) = \ldots$

- ☐ ... 0
- ☐ ... $c \cdot g'(x) \cdot h'(x)$
- ☐ ... $c \cdot g(x) \cdot h'(x) + c \cdot g'(x) \cdot h(x)$
- ☐ ... $c \cdot g'(x) \cdot h'(x) + c \cdot g(x) \cdot h(x)$

Answers

12.1 a) $f'(x) = 3 \cdot 5x^4 = 15x^4$

b) $f'(x) = (-4)\ 3x^2 = -12x^2$

c) $f'(x) = (-1)\ 10x^9 = -10x^9$

d) $f'(x) = a \cdot 3x^2 = 3ax^2$

Hint:
- a is a constant.

e) $f'(x) = n(n-1)x^{n-2}$

f) $f'(x) = 9 \cdot 3^x \cdot \ln(3)$

g) $s'(t) = \frac{g}{2}\ 2t = gt$

Hints:
- The name of the function is s, and the variable is t.
- g is a constant.

h) $S'(T) = \alpha \cdot 4T^3 = 4\alpha T^3$

i) $C'(x) = -\ 81x^2$

12.2 a) $f'(x) = 5x^4 + 6x^5$ b) $f'(x) = 10x^9 - 9x^8$ c) $f'(x) = 1 + 9x^2$

d) $f'(x) = x^3 + 6x$ e) $f'(x) = 9x^2 – 12x$ f) $f'(x) = -\ 24x^7 + 5x^4 - 3$

g) $f'(x) = 2ax + b$ h) $f'(x) = -\ 6a + 6x$ i) $f'(x) = x^2 + \frac{9}{x^4}$

j) $s'(t) = v_0 + gt$ k) $V'(r) = \frac{a}{r^2} - \frac{2b}{r^3}$ l) $C'(n) = C_0 \cdot r$

12.3 a) $f'(x) = e^x + x \cdot e^x$

b) $f'(x) = 3x^2 \cdot 3^x + x^3 \cdot 3^x \cdot \ln(3)$

c) $f'(x) = -\ 2(5x^4(x - 1) + x^5)$

d) $f'(x) = 2 \cdot e^x + (2x - 1) \cdot e^x$

e) $f'(x) = 2(-\ 3x^2 - x + 1) + (2x - 1)\ (-\ 6x - 1)$

f) $V'(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3}\right) + e^r \left(2a \cdot r + \frac{3b}{r^4}\right)$

Hints:
- V is the name of the function, and r is the variable.
- a and b are constants.

12.4 a) $f'(x) = 4\ e^{4x}$ b) $f'(x) = (-1)\ e^{-x} = -\ e^{-x}$

c) $f'(x) = -\ 2x \cdot e^{-x^2}$ d) $f'(x) = (2x - 2)\ e^{x^2 - 2x + 5}$

12.5 a) $f'(x) = e^{2x} + (x - 2)\ 2\ e^{2x} = (2x - 3)\ e^{2x}$

b) $f'(x) = -\ 2x\ e^{-x} + (2 - x^2)\ (-1)\ e^{-x} = (x^2 - 2x - 2)\ e^{-x}$

c) $f'(x) = (9x^2 - 4x + 1)\ e^{-2x} + (3x^3 - 2x^2 + x - 1)\ (-2)\ e^{-2x} = (-\ 6x^3 + 13x^2 - 6x + 3)\ e^{-2x}$

d) $P'(v) = a\left(2v\ e^{-bv^2} + v^2(-2bv)\ e^{-bv^2}\right) = 2av\ (1 - bv^2)\ e^{-bv^2}$

12.6 (see next page)

12.6	a)	12.1 b) $f'(2) = -48$	b)	12.1 g) $s'(4) = 4g$
	c)	12.2 g) $f'(-1) = -2a + b$	d)	12.5 d) $P'(1) = 2a\,(1 - b)\,e^{-b}$

12.7 a) 12.1 a)
$f''(x) = 15 \cdot 4x^3 = 60x^3$
$f'''(x) = 60 \cdot 3x^2 = 180x^2$

b) 12.2 g)
$f''(x) = 2a \cdot 1 = 2a$
$f'''(x) = 0$

c) 12.3 a)
$f''(x) = e^x + (e^x + x \cdot e^x) = (x + 2)\,e^x$
$f'''(x) = e^x + (x + 2)\,e^x = (x + 3)\,e^x$

d) 12.4 c)
$f''(x) = -2\left(e^{-x^2} + x\,(-2x)\,e^{-x^2}\right) = 2\,(2x^2 - 1)\,e^{-x^2}$
$f'''(x) = 2\left(4x\,e^{-x^2} + (2x^2 - 1)(-2x)\,e^{-x^2}\right) = 4x\,(-2x^2 + 3)\,e^{-x^2}$

12.8 a) $f''(-1) = 60\,(-1)^3 = -60$

b) $f'''(2) = 4 \cdot 2\,(-2 \cdot 2^2 + 3)\,e^{-2^2} = -\frac{40}{e^4}$

12.9 a) 4th statement

b) 3rd statement

c) 3rd statement