Exercises 9 Exponential function and equations
Exponential equations, logarithm, compound interest

Objectives

- be able to determine simple logarithms without a calculator.
- be able to solve simple exponential equations without a calculator.
- be able to calculate a common and a natural logarithm with a calculator.
- be able to apply one of the logarithmic properties in order to solve simple exponential equations.
- be able to treat specific compound interest tasks by means of logarithms.

Problems

Exponential equations, logarithm

9.1 Solve the exponential equations below **without** a calculator, i.e. find the solutions by guessing.

a) $2^x = 16$ b) $4^x = 64$ c) $5^x = 1$

d) $\left(\frac{3}{2}\right)^x = \frac{27}{8}$ e) $10^x = 1'000'000$ f) $10^x = 10$

9.2 Determine the following logarithms **without** a calculator.

a) $\log_3(27)$ b) $\log_4(16)$ c) $\log_2(128)$

d) $\log_{10}(1000)$ e) $\log_{10}(1)$

9.3 Determine the logarithms below **with** your calculator.

a) lg(1.1) b) ln(1.1) c) lg(9)

d) ln(9) e) lg(2345.67) f) ln(2345.67)

9.4 Solve the following exponential equations.

a) $10^x = 21$ b) $10^x = 256.78$ c) $10^x = 1'234'567$

9.5 Solve the exponential equations below.

a) $3^x = 99$ b) $1.01^x = 1.5$ c) $3^{x+4} = 5$

d) $5^{2x-1} = 12$ e) $1 - e^{5x} = 0.3$

Compound interest

9.6 An initial capital C_0 is invested at an annual interest rate r, compounded annually. After n years the capital amounts to C_n. Determine n.

a) $C_0 = 1000$ CHF r = 1.00% $C_n = 1220$ CHF (rounded)

b) $C_0 = 100'000$ CHF r = 2.25% $C_n = 243'519$ CHF (rounded)

9.7 How long would 10'000 CHF have to be invested at an annual interest rate of 2.5%, compounded annually, to amount to 12'000 CHF?

9.8 How long would any initial capital have to be invested at an annual interest rate of 1.25%, compounded annually, to double its value?

9.9 An initial capital of 10'000.00 CHF is invested at an unknown annual interest rate, compounded annually. After 10 years the capital amounts to 11'894.40 CHF. After how many years (from the beginning of the investment) will the capital be worth 15'000.00 CHF?

9.10 How long (rounded to years) would 1000 CHF have to be invested at a nominal annual interest rate of 2.5%, compounded daily, to earn 250 CHF total interest?

9.11 A couple needs 150'000 CHF as a down payment for a home. If they invest the 100'000 CHF they have at a nominal annual interest rate of 8%, compounded quarterly, how long will it take for the money to grow into 150'000 CHF?

9.12 The decay of the revenues for a service is given by

$$R = 50'000\ e^{-0.8x}\ \text{CHF}$$

where R are the monthly revenues in the xth month after a promotional campaign.

a) What will be the revenues in the fourth month after the campaign?

b) In which month after the campaign will the revenues have dropped below 1000 CHF, if no new campaign is initiated?

9.13 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) $\log_a(x)$ is the answer to the question ...

- ☐ ... “a to what power is x?”
- ☐ ... “x to what power is a?”
- ☐ ... “10 to what power is x?”
- ☐ ... “10 to what power is a?”

b) If $p = 2^q$ then ...

- ☐ ... $q = \left(\frac{1}{2}\right)^p$
- ☐ ... $q = \frac{p}{2}$
- ☐ ... $q = \log_2(p)$
- ☐ ... $q = \ln(2)$

c) $\ln(e) = ...$

- ☐ ... $\log_e(1)$
- ☐ ... $\log_{10}(e)$
- ☐ ... 0
- ☐ ... 1

Answers

9.1 a) x = 4 b) x = 3 c) x = 0
d) x = 3 e) x = 6 f) x = 1

9.2 a) 3

Hint:
- The expression $\log_3(27)$ is the answer to the question "3 to what power is equal to 27?"

b) 2

c) 7

d) 3

e) 0

9.3 a) 0.041... b) 0.095... c) 0.954...
d) 2.197... e) 3.370... f) 7.760...

9.4 a) $x = \log_{10}(21) = \lg(21) = 1.322...$

b) $x = \log_{10}(256.78) = \lg(256.78) = 2.409...$

c) $x = \log_{10}(1'234'567) = \lg(1'234'567) = 6.091...$

9.5 a) x = 4.182...

b) x = 40.748...

c) x = - 2.535...

Hint:
- First solve the equation for x+4.

d) x = 1.271…

e) x = - 0.071…

Hints:
- First solve for e^{5x}.
- Then solve for 5x.

9.6 $n = \frac{\lg\left(\frac{C_n}{C_0}\right)}{\lg(1+r)}$

a) n = 20

b) n = 40

9.7 $n = \frac{\lg\left(\frac{C_n}{C_0}\right)}{\lg(1+r)}$ where C_0 = 10’000 CHF, C_n = 12’000 CHF, r = 2.5%

$\Rightarrow$ n = 7.38... $\rightarrow$ 8 years

9.8 (see next page)

9.8 $C_n = C_0 (1 + r)^n$ where r = 1.25%
$C_n = 2 \cdot C_0$

$\Rightarrow \quad n = \frac{\lg(2)}{\lg(1 + r)} = 55.79... \rightarrow$ 56 years

9.9 r = 1.75% (rounded)
C_n = 15'000.00 CHF for n = 23.37... $\rightarrow$ 24 years

Hints:
- First determine the interest rate r by looking at the first 10 years (C_0 and C_{10} are known, r is unknown).
- Then determine n (C_0, C_n, and r are known, n is unknown).

9.10 $n = \frac{\lg\left(\frac{C_n}{C_0}\right)}{\lg(1 + r)}$ where C_0 = 1000 CHF, C_n = 1250 CHF, $r = \frac{2.5\%}{360}$

$\Rightarrow$ n = 3213.38… $\rightarrow$ 3214 days = 8.92… years $\rightarrow$ 9 years

9.11 $n = \frac{\lg\left(\frac{C_n}{C_0}\right)}{\lg(1 + r)}$ where C_0 = 100'000 CHF, C_n = 150'000 CHF, $r = \frac{8\%}{4} = 2\%$

$\Rightarrow$ n = 20.47... $\rightarrow$ 21 quarters = 5 years 3 months

9.12 a) R(4) = 2038 CHF (rounded)

b) x = 4.89… $\rightarrow$ 5th month

Hints:
- Determine x such that R = 1000 CHF.
- The equation 1000 CHF = 50'000 $e^{-0.8x}$ CHF has to be solved for x.

9.13 a) 1st statement

b) 3rd statement

c) 4th statement