Exercises 7 Quadratic function and equations
Quadratic function/equations, supply, demand, market equilibrium

Objectives

- know and understand the relation between a quadratic function and a quadratic equation.
- be able to solve a quadratic equation with the method of completing the square.
- be able to solve a quadratic equation by applying the quadratic formula.
- be able to solve special quadratic equations without applying the quadratic formula.
- be able to solve a quadratic equation containing a parameter.
- be able to determine the vertex form of the equation of a quadratic function out of the coordinates of the vertex and the coordinates of another point of the corresponding parabola.
- be able to determine the general form of the equation of a quadratic function out of the coordinates of three points of the corresponding parabola.
- be able to treat applied tasks in economics by means of quadratic equations or systems of quadratic equations.

Problems

7.1 Each quadratic equation can be converted into the following general form:

$$ax^2 + bx + c = 0 \qquad (a\in\mathbb{R}\setminus\{0\},\ b\in\mathbb{R},\ c\in\mathbb{R}) \qquad (*)$$

Determine the number of solutions that a quadratic equation can have, i.e. try to find out the different possible cases of the number of solutions.

Hints:
- Remember our discussion about the possible number of solutions of a linear equation.
- Compare the left hand side of the quadratic equation (*) with the general form of the equation of a quadratic function.
- Think of the graph of a quadratic function.

7.2 Solve the quadratic equations below using ...

i) ... the method of completing the square.
ii) ... the quadratic formula.

State the solution set for each equation.

a) $x^2 + 10x + 24 = 0$ b) $2x^2 - 7x + 3 = 0$

c) $x^2 + 2x + 8 = 0$ d) $x^2 - 14x + 49 = 0$

7.3 Solve the quadratic equations below using the quadratic formula. State the solution set for each equation.

a) $x^2 + 22x + 121 = 0$ b) $5x^2 + 8x - 4 = 0$

c) $5x^2 - 8x + 4 = 0$ d) $24x^2 - 65x + 44 = 0$

e) $\frac{1}{6}x^2 - \frac{5}{4}x + \frac{3}{2} = 0$ f) $- 9x^2 - 54x - 63 = 0$

7.4 Solve the equations below. State the solution set for each equation.

a) $9(x - 10) - x(x - 15) = x$ b) $3(x^2 + 2) - x(x + 9) = 11$

c) $y^3 + 19 = (y + 4)^3$ d) $\frac{9x - 8}{4x + 7} = \frac{3x}{2x + 5}$

e) $\frac{x^2}{x - 6} - \frac{6x}{6 - x} = 1$ f) $\frac{8}{x^2 - 4} + \frac{2}{2 - x} = 3x - 1$

7.5 Solve the quadratic equations below without using the quadratic formula.
State the solution set for each equation.

a) $(x + 2)(x + 5) = 0$ b) $(x - 8)(5x - 9) = 0$

c) $x^2 - 3x = 0$ d) $x^2 + 7x = 0$

e) $4x^2 - 9 = 0$ f) $100x^2 - 1 = 0$

g) $3x^2 = 27$ h) $x^2 = x$

7.6 Solve the equations below. State the solution set for each equation.

a) $(7 + x)(7 - x) = (3x + 2)^2 - (2x + 3)^2$ b) $(x - 3)(2x - 7) = 1$

c) $\frac{x - 4}{x - 5} = \frac{30 - x^2}{x^2 - 5x}$ d) $\frac{x^2 - x - 2}{2 - x} = 1$

e) $\frac{x^2 - 4}{x^2 - 4} = 0$ f) $\frac{x^2 - 4}{x^2 - 4} = 1$

7.7 The quadratic equations below contain a parameter p. Therefore, the solution set of the equations will depend on the value of this parameter.

Solve the equations for x.

a) $x^2 + x + p = 0$ b) $3x^2 + px – p = 0$

7.8 A parabola has the vertex V and contains the point P.
Determine the equation of the corresponding quadratic function both in the vertex and in the general form.

a) V(2|4) P(-1|7)

b) V(1|-8) P(2|-7)

7.9 A parabola contains the three points P, Q, and R.
Determine the equation of the corresponding quadratic function in the general form.

a) P(-4|8) Q(0|0) R(10|15)

b) P(1|-1) Q(2|4) R(4|8)

7.10 Find the equilibrium quantity and equilibrium price of a service for the given supply and demand functions f_s and f_d:

a) supply $p = f_s(q) = \left(\frac{1}{4}q^2 + 10\right)$ CHF
demand $p = f_d(q) = (86 - 6q - 3q^2)$ CHF

b) supply $p = f_s(q) = (q^2 + 8q + 16)$ CHF
demand $p = f_d(q) = (- 3q^2 + 6q + 436)$ CHF

7.11 The total costs C(x) for producing x items and the revenues R(x) for selling x items are given by

$C(x) = (2000 + 40x + x^2)$ CHF
$R(x) = 130x$ CHF

Find the break-even values of x.

7.12 The total costs C(x) for producing x items and the revenues R(x) for selling x items are given by

$C(x) = (x^2 + 100x + 80)$ CHF
$R(x) = (160x - 2x^2)$ CHF

How many items are to be produced and sold in order to achieve a profit of 200 CHF?

7.13 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) A quadratic equation ...

- ☐ ... has no solution whenever the vertex of the graph of the corresponding quadratic function is below the x-axis.
- ☐ ... always has one or two solutions.
- ☐ ... has exactly one solution if the vertex of the graph of the corresponding quadratic function is on the x-axis.
- ☐ ... can have infinitely many solutions.

b) The graph of a quadratic function ...

- ☐ ... is uniquely defined whenever the vertex and one further point of the graph are known.
- ☐ ... is a straight line if the corresponding quadratic equation has exactly one solution.
- ☐ ... is a quadratic equation.
- ☐ ... can be determined by solving a quadratic equation.

c) If the total cost function is quadratic and the total revenue function is linear ...

- ☐ ... there is always exactly one break-even point.
- ☐ ... a break-even point corresponds to a solution of a quadratic equation.
- ☐ ... no profit can be realised whenever the linear function has a positive slope.
- ☐ ... the vertex of the graph of the cost function cannot be below the x-axis.

Answers

7.1 ...

7.2
a) $S = \{-6, -4\}$ b) $S = \left\{\frac{1}{2}, 3\right\}$
c) $S = \{\ \}$ d) $S = \{7\}$

7.3
a) $S = \{-11\}$ b) $S = \left\{-2, \frac{2}{5}\right\}$
c) $S = \{\ \}$ d) $S = \left\{\frac{4}{3}, \frac{11}{8}\right\}$
e) $S = \left\{\frac{3}{2}, 6\right\}$ f) $S = \left\{-3-\sqrt{2}, -3+\sqrt{2}\right\}$

7.4
a) $S = \{5, 18\}$ b) $S = \left\{5, -\frac{1}{2}\right\}$
c) $S = \left\{-\frac{3}{2}, -\frac{5}{2}\right\}$ d) $S = \left\{2, -\frac{10}{3}\right\}$
e) $S = \{-2, -3\}$ f) $S = \left\{-\frac{5}{3}, 0\right\}$

7.5
a) $S = \{-5, -2\}$ b) $S = \left\{\frac{9}{5}, 8\right\}$
c) $S = \{0, 3\}$ d) $S = \{-7, 0\}$
e) $S = \left\{-\frac{3}{2}, \frac{3}{2}\right\}$ f) $S = \left\{-\frac{1}{10}, \frac{1}{10}\right\}$
g) $S = \{-3, 3\}$ h) $S = \{0, 1\}$

7.6
a) $S = \{-3, 3\}$ b) $S = \left\{\frac{5}{2}, 4\right\}$
c) $S = \{-3\}$ d) $S = \{-2\}$
e) $S = \{\ \}$ f) $S = \mathbb{R}\setminus\{-2, 2\}$

7.7
a)
if $p < \frac{1}{4}$: 2 solutions $x_{1,2} = \frac{-1 \pm \sqrt{1-4p}}{2}$
if $p = \frac{1}{4}$: 1 solution $x = -\frac{1}{2}$
if $p > \frac{1}{4}$: no solution $S = \{\ \}$

Hints:
- Use the quadratic formula.
- The number of solutions (2 solutions, 1 solution, no solution) of the quadratic equation will depend on whether the term under the square root is positive, negative, or equal to zero.

b)
if $p < -12$: 2 solutions $x_{1,2} = \frac{-p \pm \sqrt{p^2 + 12p}}{6}$
if $p = -12$: 1 solution $x = 2$
if $-12 < p < 0$: no solution $S = \{\ \}$
if $p = 0$: 1 solution $x = 0$
if $p > 0$: 2 solutions $x_{1,2} = \frac{-p \pm \sqrt{p^2 + 12p}}{6}$

7.8 a) $y = f(x) = \frac{1}{3}(x - 2)^2 + 4 = \frac{1}{3}x^2 - \frac{4}{3}x + \frac{16}{3}$

Hints:
- Start with the vertex form of the equation of a quadratic function.
- That equation contains three unknown parameters.
- Two parameters in the equation are the coordinates of the vertex V.
- P is a point of the graph of the quadratic function. Therefore, the coordinates of P must fulfil the equation of the quadratic function. This yields an equation which contains the remaining unknown parameter.

b) $y = f(x) = (x - 1)^2 - 8 = x^2 - 2x - 7$

7.9 a) $y = f(x) = \frac{1}{4}x^2 – x$

Hints:
- Start with the general form of the equation of a quadratic function.
- That equation contains three unknown parameters.
- P, Q, and R are points of the graph of the quadratic function. Therefore, the coordinates of P, Q, and R must fulfil the equation of the quadratic function. This yields a system of three equations in the unknown three parameters.

b) $y = f(x) = - x^2 + 8x - 8$

7.10 a) at market equilibrium: $q = 4$, $p = 14$

Hint:
- The supply and demand functions have the same values at market equilibrium.

b) at market equilibrium: $q = 10$, $p = 196$

7.11 $x_1 = 40$, $x_2 = 50$

Hint:
- The cost and revenue functions have the same values at the break-even points.

7.12 profit $P(x) = R(x) - C(x) = (- 3x^2 + 60x – 80)$ CHF = 200 CHF

$\Rightarrow$ $S = \{7.41..., 12.58...\}$

Rounding to a whole number of articles

P(7) = 193 CHF
P(8) = 208 CHF

P(12) = 208 CHF
P(13) = 193 CHF

Whether to round up or down depends on whether the profit should be as close to 200 CHF as possible or at least 200 CHF.

7.13 a) 3rd statement

b) 1st statement

c) 2nd statement