Exercises 10 Exponential function and equations
Ordinary annuity, annuity due

Objectives

- be able to calculate the present and the future value of an annuity if constant payments are made at the beginning or at the end of each compounding period.
- be able to treat specific annuity tasks.

Problems

Ordinary annuity

10.1 Find the future value of an annuity of 1300 CHF paid at the end of each year for 5 years, if interest is paid at an annual interest rate of 6%, compounded annually.

10.2 The formula

$$A_n = p\,\frac{q^n - 1}{q - 1}$$

is used for calculating the future value A_n of an ordinary annuity.

Solve the formula for p and n.

10.3 At the end of each quarter, 2500 CHF is deposited in an account that pays interest at a nominal annual interest rate of 5%, compounded quarterly. After how many quarters will the account contain 80'000 CHF?

10.4 Assume that money on a savings account pays interest at an annual interest rate of 1.5%, compounded annually. In order to have 20'000 CHF at the end of 10 years, ...

a) ... what payment must be made at the end of each year?

b) ... what amount has to be paid in at the beginning of the ten years if no more payments are made for the rest of the time?

c) Compare the answers in a) and b), and explain why the payment made in b) is smaller than the sum of the 10 payments made in a).

10.5 Two twins are 23 years old and have different investment strategies.

Suppose that twin 1 invests 2000 CHF at the end of each year for 10 years only (until age 33) in an account that pays interest at an annual interest rate of 8%, compounded annually. Suppose that twin 2 waits until turning 40 to begin investing.

How much must twin 2 put aside at the end of each year for the next 25 years in an account that also pays interest at an annual interest rate of 8%, compounded annually, in order to have the same amount as twin 1 when he turns 65?

Hints:
- Draw a diagram which shows the investment strategies of the two twins with respect to time.
- The money twin 1 has paid in by the time he turns 33 pays interest until he turns 65.

10.6 Find the initial value of an annuity if 6000 CHF can be withdrawn at the end of each 6-month period for 8 years and if interest is paid at a nominal annual interest rate of 8%, compounded semiannually.

10.7 The formula

$$A_0 = p\,\frac{q^n - 1}{q^n(q - 1)}$$

is used for calculating the initial value A_0 of an ordinary annuity.

Solve the formula for p and n.

10.8 With an initial value of 135'000 CHF, what is the size of the withdrawals that can be made at the end of each quarter for the next 10 years if money pays interest at a nominal annual interest rate of 6.4%, compounded quarterly?

10.9 A personal account earmarked as a retirement supplement contains 242'000 CHF. Suppose 200'000 CHF is used to establish an annuity that pays interest at a nominal annual interest rate of 6%, compounded quarterly, and pays 4500 CHF at the end of each quarter. How long will it be until the account balance is 0 CHF?

Annuity due

10.10 The two formulae

$$A_n = pq\,\frac{q^n - 1}{q - 1} \qquad \text{and} \qquad A_0 = p\,\frac{q^n - 1}{q^{n-1}(q - 1)}$$

are used for calculating the future value A_n or the initial value A_0 of an annuity due.

Solve both formulae for p and n.

10.11 Find the future value of an annuity due where 100 CHF are paid in each quarter for 2.5 years and where interest is paid at a nominal annual interest rate of 12%, compounded quarterly.

10.12 How much must be deposited at the beginning of each year in an account that pays interest at an annual interest rate of 8%, compounded annually, so that the account will contain 24'000 CHF at the end of 5 years?

10.13 An account that pays interest at a nominal annual interest rate of 5%, compounded quarterly, contains 80'000 CHF at the beginning. If 2500 CHF is withdrawn at the beginning of each quarter, after how many quarters will the account contain 0 CHF?

10.14 What amount must be set aside now to generate payments of 50'000 CHF at the beginning of each year for the next 12 years if money pays interest at an annual interest rate of 5.92%, compounded annually?

10.15 A year-end bonus of 25'000 CHF will generate how much money at the beginning of each month for the next year, if it can be invested at a nominal annual interest rate of 6.48%, compounded monthly?

Miscellaneous problems

10.16 Mr. Gordon plans to invest 300 CHF at the end of each month in an account that pays interest at a nominal annual interest rate of 9%, compounded monthly. After how many months will the account be worth 50'000 CHF?

10.17 (see next page)

10.17 Grandparents plan to open an account on their grandchild's birthday and contribute each month until she goes to college. How much must they contribute at the beginning of each month in an investement that pays interest at a nominal annual interest rate of 12%, compounded monthly, if they want the balance to be 180'000 CHF at the end of 18 years?

10.18 An insurance settlement of 750'000 CHF must replace somebody's income for the next 40 years. What income will this settlement provide at the end of each month if it is invested in an annuity that pays interest at a nominal annual interest rate of 8.4%, compounded monthly?

10.19 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) In an ordinary annuity scheme ...

- ☐ ... money is always paid in or withdrawn once per year.
- ☐ ... money is paid in or withdrawn at the beginning of each period.
- ☐ ... the value of the annuity grows or decays exponentially.
- ☐ ... no payments are made during a compounding period.

b) In an ordinary annuity scheme interest is compounded monthly. If 100 CHF are paid in each month it can be concluded that the value of the annuity after one year is ...

- ☐ ... 1200 CHF.
- ☐ ... 1320 CHF if the nominal annual interest rate is 10%.
- ☐ ... less than 1320 CHF if the nominal annual interest rate is 10%.
- ☐ ... less than 1200 CHF.

c) Assume an initial capital of 1000 CHF. In an annuity due scheme (annual interest rate = 1%, compounded annually) a constant amount of money should be withdrawn 10 times at the beginning of each year. Therefore, the annual withdrawals ...

- ☐ ... must not be greater than 100 CHF.
- ☐ ... must be exactly 100 CHF.
- ☐ ... could be twice as high if the annual interest rate equalled 2%.
- ☐ ... could be greater in an ordinary annuity scheme.

Answers

10.1 $A_n = p\,\frac{q^n - 1}{q - 1}$ where p = 1300 CHF, q = 1 + 6% = 1.06, n = 5

$\Rightarrow$ A_5 = 7328.22 CHF (rounded)

10.2 see formulary

10.3 $n = \frac{\lg\left(\frac{A_n(q-1)}{p} + 1\right)}{\lg(q)}$ where A_n = 80'000 CHF, p = 2500 CHF, $q = 1 + \frac{5\%}{4}$

$\Rightarrow$ n = 27.08... $\rightarrow$ 28 quarters = 7 years

10.4 a) Ordinary annuity

$p = \frac{A_n(q-1)}{q^n - 1}$ where A_n = 20'000 CHF, q = 1 + 1.5%, n = 10

$\Rightarrow$ p = 1868.70 CHF (rounded up)

b) Compound interest

$C_0 = \frac{C_n}{q^n}$ where C_n = 20'000 CHF, q = 1 + 1.5%, n = 10

$\Rightarrow$ C_0 = 17'233.35 CHF (rounded up)

c) The payment in b) pays interest throughout the10 years.
The single payments in a) do not pay interest throughout the 10 years.

10.5 Twin 1: Ordinary annuity (from age 23 to age 33)

$A_n = p\,\frac{q^n - 1}{q - 1}$ where p = 2000 CHF, q = 1 + 8%, n = 10

$\Rightarrow$ A_{10} = capital at the age of 33 = 28'973.12 CHF (rounded)

Compound interest (from age 33 to age 65)

$C_n = C_0\, q^n$ where $C_0 = A_{10}$, q = 1 + 8%, n = 32

$\Rightarrow$ C_{32} = capital at the age of 65 = 340'059.97 CHF (rounded)

(C_{32} = capital of twin 2 at the age of 65)

Twin 2: Ordinary annuity (from age 40 to age 65)

$p = \frac{A_n(q-1)}{q^n - 1}$ where $A_n = C_{32}$ (twin 1) = 340'059.97 CHF, q = 1 + 8%, n = 25

$\Rightarrow$ p = 4651.61 CHF (rounded)

10.6 $A_0 = p\,\frac{q^n - 1}{q^n(q - 1)}$ where p = 6000 CHF, $q = 1 + \frac{8\%}{2}$, n = 8·2 = 16

$\Rightarrow$ A_0 = 69'913.77 CHF (rounded)

10.7 see formulary

10.8 $p = \frac{A_0\, q^n(q-1)}{q^n - 1}$ where A_0 = 135'000 CHF, $q = 1 + \frac{6.4\%}{4}$, n = 10·4 = 40

$\Rightarrow$ p = 4595.46 CHF (rounded)

10.9 $n = \frac{\lg\left(\frac{p}{p - A_0(q-1)}\right)}{\lg(q)}$ where $A_0 = 200'000$ CHF, $p = 4500$ CHF, $q = 1 + \frac{6\%}{4}$

$\Rightarrow$ $n = 73.78...$ $\rightarrow$ 73 quarters (less than 4500 CHF at the end of the 74th quarter)

10.10 see formulary

10.11 $A_n = pq\,\frac{q^n - 1}{q - 1}$ where $p = 100$ CHF, $q = 1 + \frac{12\%}{4}$, $n = 2.5 \cdot 4 = 10$

$\Rightarrow$ $A_{10} = 1180.78$ CHF (rounded)

10.12 $p = \frac{A_n(q-1)}{q(q^n - 1)}$ where $A_n = 24'000$ CHF, $q = 1 + 8\%$, $n = 5$

$\Rightarrow$ $p = 3787.92$ CHF (rounded)

10.13 $n = \frac{\lg\left(\frac{pq}{pq - A_0(q-1)}\right)}{\lg(q)}$ where $A_0 = 80'000$ CHF, $p = 2500$ CHF, $q = 1 + \frac{5\%}{4}$

$\Rightarrow$ $n = 40.46...$ $\rightarrow$ 40 quarters (less than 2500 CHF at the beginning of the 41st quarter)

10.14 $A_0 = p\,\frac{q^n - 1}{q^{n-1}(q - 1)}$ where $p = 50'000$ CHF, $q = 1 + 5.92\%$, $n = 12$

$\Rightarrow$ $A_0 = 445'962.23$ CHF (rounded)

10.15 $p = \frac{A_0\, q^{n-1}(q-1)}{q^n - 1}$ where $A_0 = 25'000$ CHF, $q = 1 + \frac{6.48\%}{12}$, $n = 1 \cdot 12 = 12$

$\Rightarrow$ $p = 2145.59$ CHF (rounded)

10.16 Ordinary annuity

$n = \frac{\lg\left(\frac{A_n(q-1)}{p} + 1\right)}{\lg(q)}$ where $A_n = 50'000$ CHF, $p = 300$ CHF, $q = 1 + \frac{9\%}{12}$

$\Rightarrow$ $n = 108.52...$ $\rightarrow$ 109 months (= 9 years 1 month)

10.17 Annuity due

$p = \frac{A_n(q-1)}{q(q^n - 1)}$ where $A_n = 180'000$ CHF, $q = 1 + \frac{12\%}{12}$, $n = 18 \cdot 12 = 216$

$\Rightarrow$ $p = 235.16$ CHF (rounded)

10.18 Ordinary annuity, income = monthly payment p

$p = \frac{A_0\, q^n(q-1)}{q^n - 1}$ where $A_0 = 750'000$ CHF, $q = 1 + \frac{8.4\%}{12}$, $n = 40 \cdot 12 = 480$

$\Rightarrow$ $p = 5441.23$ CHF (rounded)

10.19 a) 4th statement

b) 3rd statement

c) 4th statement